SCET and Collider Physics — Tutorial 4

Wilson lines, light-cone vectors — 26. 5. 2016

1. Consider two light-cone vectors n_μ and $\bar{n}_\mu$, with $\bar{n} \cdot n = 2$. Show that the operators

$$P_+ = \frac{\not{n}\not{\bar{n}}}{4}\,, \qquad\qquad P_- = \frac{\not{\bar{n}}\not{n}}{4}\,,$$

are projection operators with $P_+ + P_- = 1$.

2. Use the projection operators $P_\pm$ to split the quark field into two components

$$\psi(x) = \xi(x) + \eta(x) = P_+\psi(x) + P_-\psi(x)$$

Show that

(a) $\not{n}\,\xi(x) = 0$,

(b) $\bar{\xi}(x)\,\xi(x) = 0$,

(c) $\bar{\xi}(x)\,\not{D}_\perp \xi(x) = 0$,

(d) $\bar{\xi}(x)\gamma^\mu\xi(x) = n^\mu\,\bar{\xi}(x)\frac{\not{\bar{n}}}{2}\xi(x)$.

3. Consider the QED Wilson line

$$[z, y] = \exp\left[-ie\int_{\mathcal{C}} dx_\mu A^\mu(x)\right]\,,$$

where the curve $\mathcal{C}$ goes from y to z. Show that under a gauge transformation $V(x) = \exp(i\alpha(x))$ the Wilson line transforms as

$$[z, y] \to V(z)[z, y]V^\dagger(y)\,.$$

4. Consider the QCD Wilson line along some path $x^\mu \equiv x^\mu(s)$ with starting point $x^\mu(0) = y^\mu$. It is defined as

$$[x, y] = \mathbf{P}\exp\left[ig\int_0^s ds\,\boldsymbol{F}(s)\right]\,, \tag{1}$$

where the exponent is the color matrix

$$\boldsymbol{F}(s) \equiv \frac{dx^\mu}{ds}A^b_\mu\left(x(s)\right)t^b\,. \tag{2}$$

and $\mathbf{P}$ denotes path-ordering which enforces that the matrices appearing at later point on the path arise on the left of earlier ones, for example

$$\mathbf{P}\left[\boldsymbol{F}(s_1)\boldsymbol{F}(s_2)\right] \equiv \boldsymbol{F}(s_2)\boldsymbol{F}(s_1) \text{ for } s_2 > s_1\,. \tag{3}$$

Show that the derivative of the Wilson line along the path vanishes,

$$\frac{dx^\mu}{ds} D_\mu \, [x, y] = \left(\frac{d}{ds} - ig\boldsymbol{F}(s) \right) [x(s), y] = 0\,.$$

To derive this property, expand the path-ordered exponential in a Taylor series and show that the n-th order term can be rewritten as an ordered integration

$$\frac{(ig)^n}{n!} \int_0^s ds_1 \int_0^s ds_2 \cdots \int_0^s ds_n \, \mathbf{P} \left\{ \boldsymbol{F}(s_1)\boldsymbol{F}(s_2) \cdots \boldsymbol{F}(s_n) \right\} = (ig)^n \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \, \boldsymbol{F}(s_1)\boldsymbol{F}(s_2) \cdots \boldsymbol{F}(s_n)$$