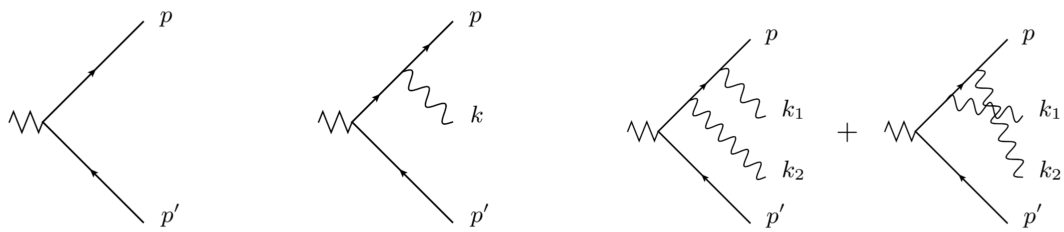
1. The following Feynman diagrams



describe the production of an $e^-(p)e^+(p')$ pair from a virtual photon (first diagram) together with one (second diagram) or two soft photons (last two diagrams) emitted from the outgoing electron. The result for the first diagram is $\mathcal{M}_0 = -ie\bar{u}(p)\gamma^\mu v(p')$. We consider the diagrams in the soft limit where the outgoing photons are soft, i.e. k, k_1, $k_2 \to 0$.

   (a) Show that the leading term in the expansion around the soft limit is

$$\mathcal{M}_1(k) = \mathcal{S}_1(k)\mathcal{M}_0 = e\frac{\varepsilon \cdot p}{k \cdot p}\mathcal{M}_0\,, \tag{1}$$

   for the emission of a single photon from the electron (second diagram), where ε is the polarization vector of the photon.

   (b) Show that the result for the emission of two photons from the electron (i.e. the sum of the last two diagrams) is

$$\mathcal{M}_2(k_1,k_2) = \mathcal{S}_2(k_1,k_2)\mathcal{M}_0 = \frac{e^2}{2}\frac{\varepsilon_1 \cdot p}{k_1 \cdot p}\frac{\varepsilon_2 \cdot p}{k_2 \cdot p}\mathcal{M}_0 = \frac{1}{2!}\mathcal{S}_1(k_1)\mathcal{S}_1(k_2)\mathcal{M}_0\,, \tag{2}$$

   where ε is the polarization vector of the photon.

2. Consider the following QED Wilson line operator

$$S_v(y) = \exp\left[-ie\int_0^\infty ds\, v \cdot A(y+vs)\, e^{-\delta s}\right].$$

The factor $e^{-\delta s}$ was inserted to ensure that the integral converges for $s \to \infty$ and one takes $\delta \to 0$ after evaluating the matrix elements of the operator.

Compute the matrix elements of the operator $S_v(0)$ with final state photons

$$\mathcal{S}_1(k) = \langle k, \varepsilon | S_v(0) | 0 \rangle\,,$$
$$\mathcal{S}_2(k_1,k_2) = \langle k_1, \varepsilon_1;\, k_2, \varepsilon_2 | S_v(0) | 0 \rangle$$

and show that the results are identical to the one obtained in the previous exercise if $v^\mu \propto p^\mu$, i.e. if the reference vector v^μ is chosen along the direction of the electron momentum.

3. Derive the momentum-space Feynman rules for the interaction terms in the Euler-Heisenberg Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{m_e^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{c_2}{m_e^4}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} + \mathcal{O}\left(\frac{E_\gamma^6}{m_e^6}\right) .$$

It suffices to consider the term proportional to c_1. In principle, one should symmetrize the Feynman rule in the external lines, but for the purpose of the exercise, is good enough to give a result without symmetrization.

4. Using power-counting arguments, we showed in the lecture that the $\gamma\gamma \to \gamma\gamma$ cross section is suppressed by $\alpha^4 E_\gamma^6/m_e^8$ at low photon energies. What is the relevant suppression factor for the $\gamma\gamma \to \gamma\gamma\gamma\gamma$ process?