Warm up: $\gamma\gamma$ and e^-e^- scattering

As a start, we will consider SET (soft photon effective theory). This is quite a bit simpler than SCET, but will allow us to discuss a few basic EFT concepts as well as some ingredients which arise in modern EFTs such as SCET.

We'll consider processes which involve only soft photons with $E_\gamma \ll m_e$. In a first step, we'll study $\gamma\gamma$ scattering. The corresponding EFT is called Euler-Heisenberg theory and is <u>the</u> classic example of an EFT.

Then we turn to $e^-e^- \to e^-e^-$ + "soft photons". The relevant EFT is called heavy electron EFT (typically it is used for quarks and called HQET).

The result of our derivation will be two factorization theorems. First of all, we will show that the amplitude for $\gamma\gamma$ factorizes in the form

$$\mathcal{A}(\gamma\gamma \to \gamma\gamma) = \sum_{i=1}^{2} C^{(i)}(m_e) \cdot S^{(i)}(E_\gamma, \cos\theta)$$

$\uparrow$ Hard func. $\qquad$ $\uparrow$ Soft func.

with $\quad S^{(i)} = \langle \gamma\gamma | O^{(i)} | \gamma\gamma \rangle$

Then, we will show that for $e^+e^- \to e^+e^- + X_\gamma^{soft}$, we end up with

$$\mathcal{A}(\underset{1}{e^+}\underset{2}{e^-} \to \underset{3}{e^+}\underset{4}{e^-} + X_\gamma^{soft})$$

!

$$= C(m_e, \{\underline{v}_i\}, \mu)\ S(\{\underline{v}\}, \{\underline{p}_\gamma\}; \mu)$$

$\{\underline{v}\} = \{v_1, \dots, v_4\}$ ... velocities of the particles.

$$S = \langle X_\gamma | S_3^+ S_4^+ S_1 S_2 | 0 \rangle$$

see later!
not local operators

This allows us to derive classical results of Yennie, Frautschi and Suura '61 in a simple way.

<u>Euler - Heisenberg Theory</u>

consider processes involving only soft photons, e.g.

For $E_\gamma \ll m_e$ e^+e^- are highly virtual and we should be able to integrate them out:

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(A_\mu)$$

To ensure that we reproduce QED, we allow for the most general Lagrangian.

→ ∞ many operators! Order them by their dimension.

$\underline{d=2}$

$\cancel{A_\mu A^\mu}$ not gauge inv!

$\underline{d=4}$

$F_{\mu\nu} F^{\mu\nu}$

$\underline{d=6}$

$F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\mu = 0$ in QED!

$O_6 = (\partial_\mu F^{\rho\mu})(\partial^\nu F_{\rho\nu})$ + 2 others which are total derivatives

Classical EOM: $\partial_\mu F^{\rho\mu} = j^\rho = 0$

→ O_6 does not contribute to on-shell matrix elements. (can be eliminated by field redef.)

First interesting operators arise at

$\underline{d=8}$ $\quad O_8^{(a)} = (F_{\mu\nu}F^{\mu\nu})^2$

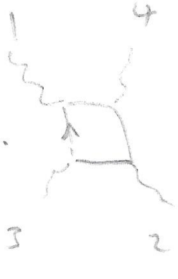
$$O_8^{(b)} = F^{\mu}{}_{\nu} F^{\nu}{}_{\rho} F^{\rho}{}_{\sigma} F^{\sigma}{}_{\mu}$$

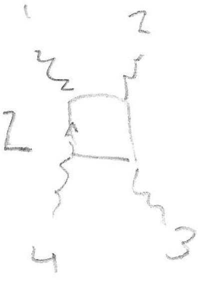
$$\mathcal{L}_{eff}^{EH} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + C^{(a)} O_8^{(a)} + C^{(b)} O_8^{(b)} + \dots$$

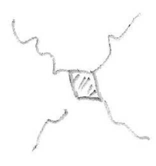
Dimensional analysis $\quad C^{(i)} \sim \frac{1}{m_e^4}$

Matching:

QED: $\quad \mathcal{A} = 2\cdot$ 

$+ \; 2\cdot$ 

$+ \; 2$ 

EFT: $\quad \mathcal{A} = C^{(a)}$  $+ \; C^{(b)}$ 

computing $\mathcal{A}$ for different polarizations yields:

$$C^{(a)} = -\frac{\pi}{36}\frac{\alpha^2}{m_e^4} \qquad C^{(b)} = +\frac{7\pi}{90}\frac{\alpha^2}{m_e^4}$$

However, even without a detailed computation, we can estimate

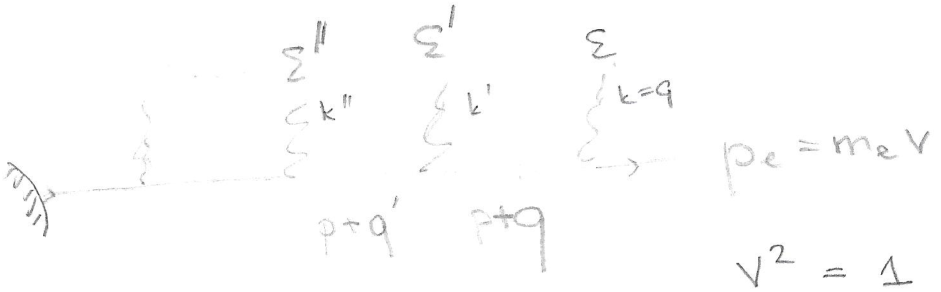
$$\sigma_{\gamma\gamma\to\gamma\gamma} \sim \underbrace{\left(\frac{\alpha^2}{m_e^4}\right)^2}_{C_8^{(i)}} \underset{\substack{\uparrow \\ \sigma \,\sim\, L^2 \,\sim\, \frac{1}{E^2}}}{E_\gamma^6}$$

Strongly suppressed!

Heavy-particle EFT

Due to fermion number conservation, we cannot always integrate out the fermions, even at low energies.

Consider a single electron, surrounded by soft photons



$$v^2 = 1$$

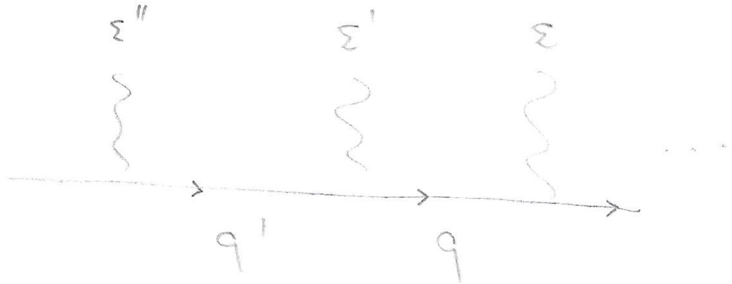
Propagator can be expanded

$$\Delta_F(p+q) = i\,\frac{\not p + \not q + m}{(p+q)^2 - m^2 + i\varepsilon} = i\,\frac{\not p + m}{2p\cdot q + i\varepsilon} + O\left(\frac{1}{m}\right)$$

$$= i\,\frac{1+\not v}{2}\cdot\frac{1}{v\cdot q + i\varepsilon} = P_+\,\frac{i}{v\cdot q}$$

Note that $P_+^2 = P_+$; $\not v P_+ = P_+$

$$P_+ \not\varepsilon P_+ = P_+\,\varepsilon\cdot v$$



$$\ldots\; P_+\,\frac{i}{v\cdot q + i\varepsilon}(-ie\,\varepsilon'\cdot v)\; P_+\,\frac{i}{v q' + i\varepsilon}(-ie\,\varepsilon''\cdot v)\,)\ldots$$

Can we obtain this from $\mathcal{L}_{eff}$?!

Yes: $\mathcal{L}_{eff}^{(4)} = \bar{h}_v(x)\, i v\cdot D\, h_v(x) + O\left(\frac{1}{m_c}\right)$

where $P_+ h_v = h_v$

$\rightarrow \not v h_v = h_v$

[Note that $\psi_e(x) = e^{-imvx}\left(P_+ h_v(x) + \ldots\right)$

Interesting difference to EH: we need dedicated field $h_v(x)$, which depends on reference vector v^μ to describe low-E e^- propagating in v^μ direction.

Propagator has single pole: $\frac{1}{v\cdot k + i\varepsilon}$

no anti-particle! (it was "integrated out")
no loops.

The general construction of $\mathcal{L}_{eff}$ then proceeds as before: write down most general set of operators of given dimension:

$$\rightarrow \quad \mathcal{L}_{eff} = \bar{h}_v\, i v D\, h_v - c_{kin} \bar{h}_v \frac{D_\perp^2}{2m} h_v$$

$$+ \frac{c_{mag}}{4m} \bar{h}_v e F^{\mu\nu} \sigma_{\mu\nu} h_v + O\left(\frac{1}{m^2}\right)$$

$$\left[D_\perp^\mu = D^\mu - v\cdot D\, v^\mu \right] \quad + \mathcal{L}^{EH}$$

Matching yields (a) $C_{kin} = 1$

(b) $C_{mag} = 1 + \frac{\alpha}{2\pi}$

anomalous mag. moment.

(a) Holds to all orders! Consequence of Lorentz invariance: $E = \vec{p}^{\,2}/2m + O(p^4)$

(set $v^\mu = (1, \vec{0})$)

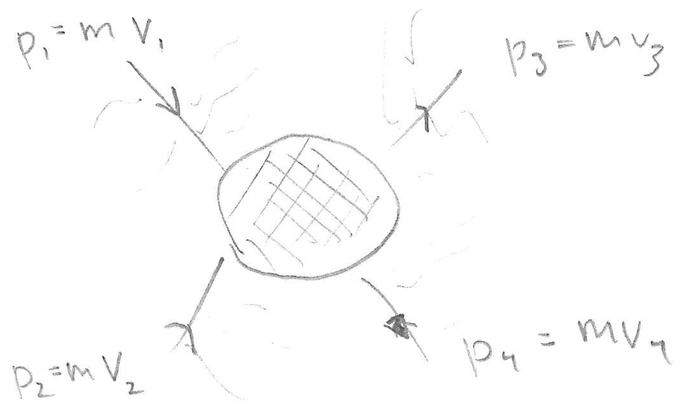
The main use of $\mathcal{L}_{eff}$ has been in flavor physics, to understand properties of B and D mesons (In this case, the gluon field is the relevant gauge field).

However, we will now use it to analyze

$e^- e^- \to e^- e^-$ + "soft photons".

If a photon is soft enough, it will not be detected, so <u>any</u> physical cross section must allow for soft photons in the final state. In fact, scattering cross sections w/o soft photons suffer from IR divergencies at higher order!

To analyze the scattering process in our EFT, we need a dedicated field $h_i = h_{v_i}$ for each particle:



$$\mathcal{L}_{eff} = \sum_{i=1}^{4} \bar{h}_i \, v_i \cdot D \, h_i - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$+ \text{"interaction terms"}$$

The lowest order interaction terms have the form

Dirac index

$$\Delta \mathcal{L} = C_{\alpha\beta\gamma\delta}(v_1, v_2, v_3, v_4)\, h_1^{\alpha}\, h_2^{\beta}\, \bar{h}_3^{\gamma}\, \bar{h}_4^{\delta}$$

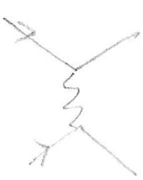
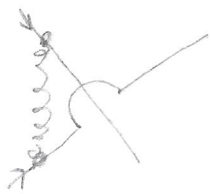
This is slightly clumsy: we now that $\Delta\mathcal{L}$ is a scalar. Can write

$$\Delta \mathcal{L} = \sum_i C_i(v_1, v_2, v_3, v_4)\, \bar{h}_3 \Gamma_i h_1\, \bar{h}_4 \Gamma_i h_2$$

where C_i are scalar functions, but clumsy will be good enough! $\Gamma_i = \mathbb{1} \otimes \mathbb{1},\ \gamma^{\mu} \otimes \gamma_{\mu}, \ldots$

Now let's do the matching computation:

QED

$\mathcal{A}$ =  +  + ...

EFT

$$\mathcal{A} = \sum_i C_i(v_1, v_2, v_3, v_4)\, \bar{u}(v_3)\Gamma_i u(v_1)\, \bar{u}(v_4)\Gamma_i u(v_2)$$

on-shell scattering amplitudes are Wilson coefficients!

Now consider the object

$$S_i(x) = \exp\left[-ie\int_0^\infty ds\, v_i\cdot A(x+s v_i)\right]$$

"Wilson line"

This fulfills

$$v_i\cdot D\, S_i(x) = 0.$$ (see tutorial)

and perform a field redefinition

$$h_i(x) = S_i(x)\, h_i^{(0)}(x)$$

$$\rightarrow \quad \bar{h}_i(x)\, i v_i\cdot D\, h_i(x)$$

$$= \bar{h}_i^{(0)} S^{\dagger}(x)\, i v_i\cdot D\, S_i h_i^{(0)}(x)$$

$$= \bar{h}_i^{(0)} S_i^{\dagger} S_i\, i v\cdot\partial\, h^{(0)} = \bar{h}_i^{(0)}\, i v\partial\, h_i^{(0)}$$

The field $h_2^{(0)}$ no longer interacts with photons at leading power!

$$\Delta \mathcal{L} = \sum_i C_i \, \bar{h}_3^{(0)} S_3^{\dagger} \Gamma_i S_1 h_1^{(0)} \bar{h}_4^{(0)} S_4^{\dagger} \Gamma_i S_2 h_2^{(0)}$$

Now compute $e^-(p_1)\, e^-(p_2) \to e^-(p_3)\, e^-(p_4) + \underbrace{\gamma(k_1) + \dots + \gamma(k_n)}_{X_s}$

in the EFT. We get

$$\mathcal{A}(e^- e^- \to e^- e^- + X_s) =$$

$$\sum_i C_i \, \bar{u}(v_3) \Gamma_i u(v_1) \, \bar{u}(v_4) \Gamma_i u(v_2)$$

$$\cdot \langle \gamma(k_1) \dots \gamma(k_n) | S_3^{\dagger} S_1 S_4^{\dagger} S_2 | 0 \rangle$$

$$= \mathcal{A}(e^- e^- \to e^- e^-) \cdot \langle \gamma(k_1) \dots \gamma(k_n) | S_3^{\dagger} S_1 S_4^{\dagger} S_2 | 0 \rangle$$

This is now a nontrivial example of an all-order factorization theorem!

It turns out that matrix elements of Wilson lines have extremely simple properties: they exponentiate, i.e. the multiple emissions can be obtained by exponentiating amplitudes with one emission per leg. In the tutorial we'll see this property in the simplest example.

There are several interesting aspects of our EFT treatment:

* Reference vectors
* Multiple field for the same particle
* Expansion of diagrams (How about loops ?)

In the next lecture, we'll discuss the "method of regions", a technology which provides the mathematical basis for our treatment.