Symmetry Breaking and Solitons

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Dirac Monopole 1931

- Quantization of electric / magnetic charge
- Magnetic monopole is point-singularity
- Wave front of electron in background field is section of complex line-bundle L over $\mathbb{R}^3 - 0 \sim S^2$
- Topological invariant $= c_1(L)$
- (degree of map $S^1(equator) \to U(1)$)

't Hooft - Polyakov Monopole 1974

- $SU(2)$ - gauge field on $\mathbb{R}^3$
- symmetry broken to $U(1)$
  by (adjoint) Higgs field ϕ , $|\phi| \to 1$ at ∞
- Magnetic charge $\mu = c_1(L)$ (L Higgs line-bundle over S^2_∞)
- ($=$ degree ϕ_∞ : $S^2(space) \to S^2$ (Lie algebra))
- Minimum of Energy function (Yang-Mills-Higgs) given by (smooth) soliton solution

Skyrmions 1962

- nucleus modelled by solitions of non-linear field of pions (mass-less, Goldstone bosons)
- $f : \mathbb{R}^3 \to SU(2) \qquad f(x) \to 1 \ \ at \ \ \infty$
- $deg \ \ f : (\mathbb{R}^3 \cup \infty) \to S^3(group)$
  represents Baryon number B $= c_2$
- $f = \sigma + i\boldsymbol{\pi}.\boldsymbol{\tau} \qquad \boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ pion fields
  $\boldsymbol{\tau}$ = Pauli matrices $\quad (\sigma^2 + \boldsymbol{\pi}.\boldsymbol{\tau} = 1)$

Relation between monopoles and skyrmions ?

- Manton (1980s)
  Both are soliton models of particles in $\mathbb{R}^3$
  (monopoles, baryons)
  with topologically defined "particle number"

Instantons 1975

- Solitons of pure Yang-Mills theory in $\mathbb{R}^4$: $SU(2)$
- Given by Minimum of Yang-Mills action
  instanton number $I = c_2$
- (a) solutions (with $8I$ parameters) by
  ADHM (depends on Penrose twistor theory)
- (b) on general 4-manifolds exploited by Donaldson

Instantons, Monopoles and Skyrmions

- (a) <u>dimensional</u> <u>reduction</u> $\mathbb{R}^4 \to \mathbb{R}^3$

  $\mathbb{R}^1$ - invariant instanton $\longleftrightarrow$ BPS monopole
- self-duality equation $\longleftrightarrow$ Bogomolny equation

  $*F = F$ $\qquad\qquad$ $*F = D\phi$

  ("$I = \infty$") Nahm transform $\to$ ADHMN (1982)
- (b) Parallel transport $\mathbb{R}^4 \to \mathbb{R}^3$
- Instanton $\longrightarrow$ Skyrme field $\qquad I = B$

  Atiyah-Manton (1989)

Hyperbolic 3-space 1984

- S^1 - invariant instantons $\leftrightarrow$ monopoles on H^3_k
  "weight" k (curvature $-\frac{1}{k^2}$)
- $I = 2k\mu$
  (for $k = \frac{1}{2}, I = \mu$)

Massive Pions 2004

- Identify $\mathbb{R}^3$ with H^3_k by fixing origin
  and Euclidean length $\rightarrow$ Hyperbolic length along rays through origin. Use radial gauge
- monopole on H^3_k $\rightarrow$ Skyrmion with massive pions
  with pion mass $m = 2k$
  (exponential decay $\sim e^{-mr}$)
  $f = exp(\phi)$
  Exponentiating Higgs

References

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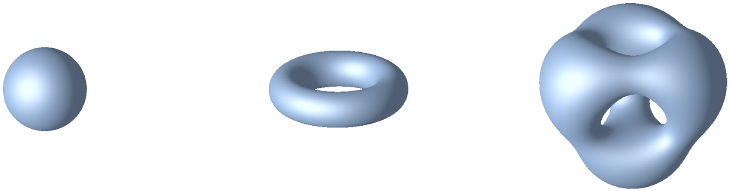
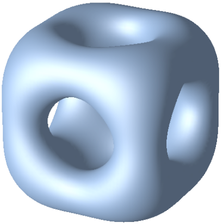
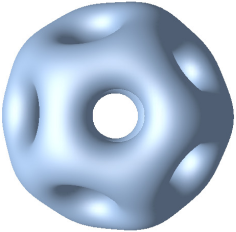
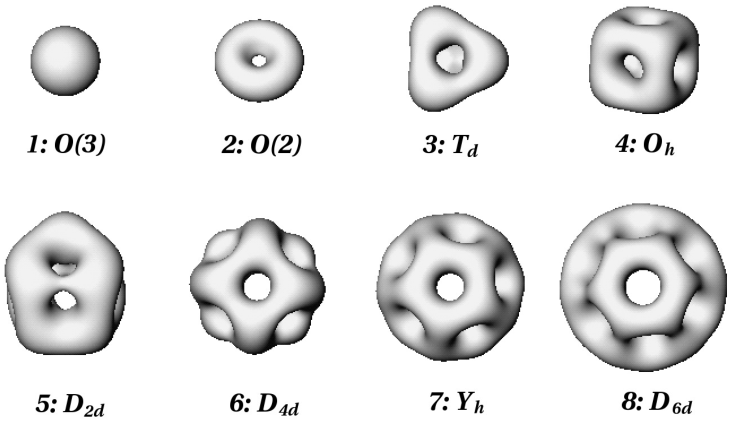
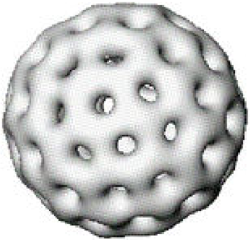
- 3. Manton and Sutcliffe, *Platonic hyperbolic monopoles* [hep-th] 11 July 2012 arXiv:1207-2636
- 4. Atiyah, *Magnetic monopoles in hyperbolic space*, Proc. Bombay Colloquium 1984, OUP(1987) 1-34

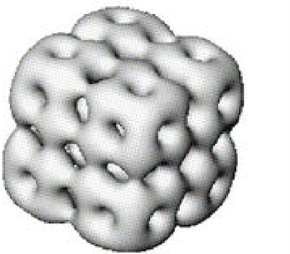







1: $O(3)$
2: $O(2)$
3: T_d
4: O_h
5: D_{2d}
6: D_{4d}
7: Y_h
8: D_{6d}





Mass Production