TENSOR NETWORKS FOR STEADY STATES

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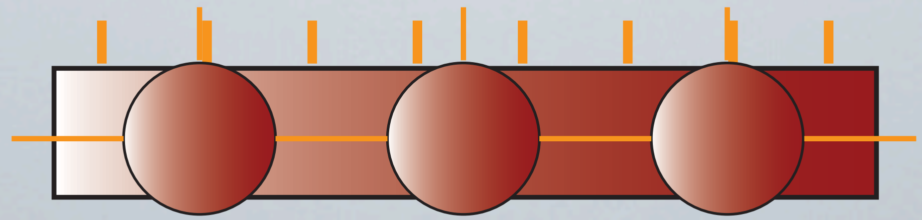
MPS



$$|\Psi\rangle = \sum_{i_1 \dots i_N} c_{i_1 \dots i_N} |i_1 \dots i_N\rangle$$

$$|\Psi\rangle = \sum_{i_1 \dots i_N} \mathrm{tr}(A_1^{i_1} A_2^{i_2} \dots A_N^{i_N}) |i_1 \dots i_N\rangle$$

Two main algorithms:

variational minimization of energy

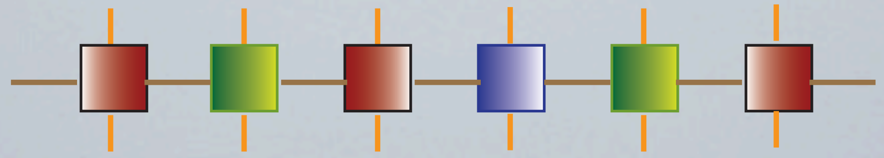
→ approximate ground state as TNS

apply local operators → simulate time evolution

imaginary time → ground state

BASIC PROBLEMS

HAMILTONIAN



find ground states

produce an ansatz for the state

→ variational search

typically faster and more precise

→ imaginary time evolution

preserves symmetries

time-dependence → real time evolution

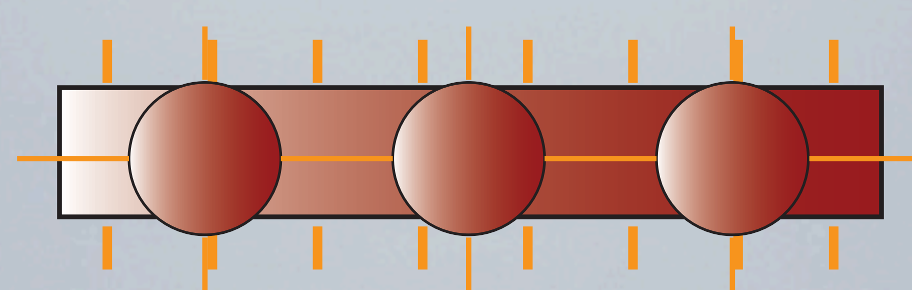
will work for short times

FOR MIXED STATES...

MIXED STATES

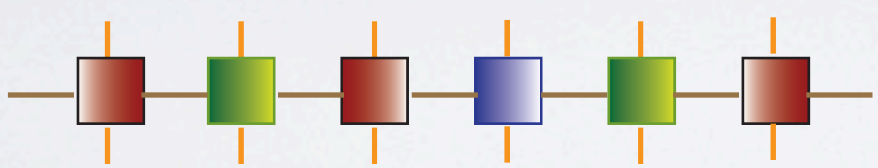
- MPO = Matrix Product Operator

Same kind of ansatz for operators



$$\hat{M} = \sum_{i_1,j_1\ldots i_N,j_N} \mathrm{tr}(M_1^{i_1 j_1} M_2^{i_2 j_2} \ldots M_N^{i_N j_N}) |i_1 \ldots i_N\rangle\langle j_1 \ldots j_N|$$

$$|\Psi\rangle = \sum_{i_1\ldots i_N} \mathrm{tr}(A_1^{i_1} A_2^{i_2} \ldots A_N^{i_N}) |i_1 \ldots i_N\rangle$$



Routinely used for H and $U(t)$

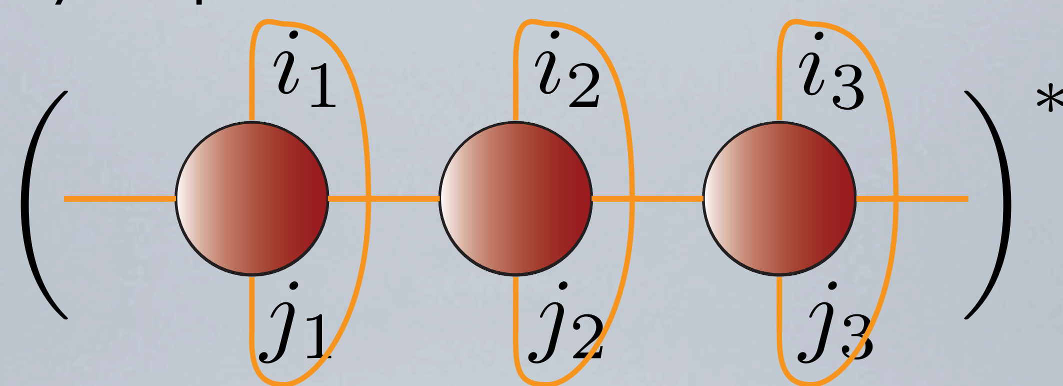
Verstraete et al., PRL 2004

Pirvu et al., NJP 2010

MIXED STATES

- MPDO = Matrix Product Density Operator

Use for density operators



need some properties

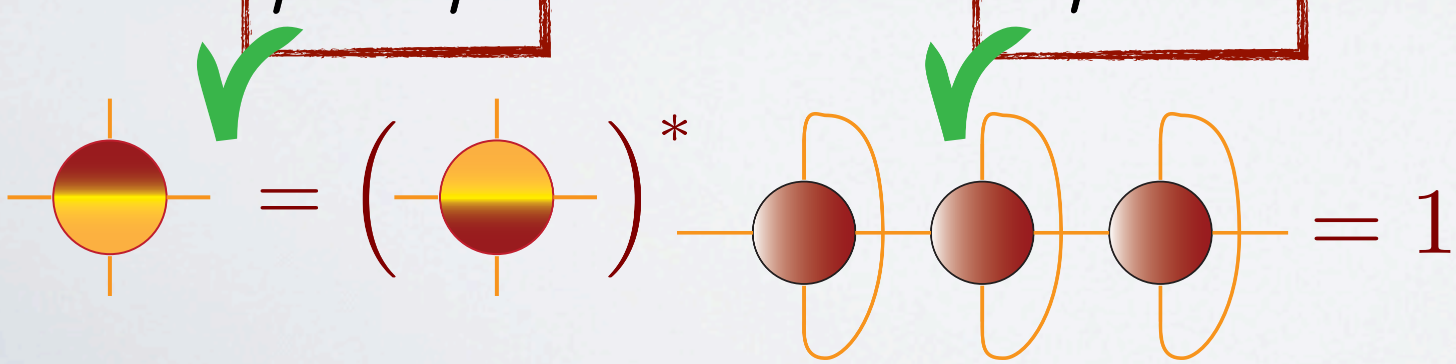
$$\rho = \sum_{i_1, j_1 \ldots i_N, j_N} \mathrm{tr}(M_1^{i_1 j_1} M_2^{i_2 j_2} \ldots M_N^{i_N j_N}) |i_1 \ldots i_N\rangle\langle j_1 \ldots j_N|$$

can we impose them *locally*?

$\rho = \rho^\dagger$ ✓

$\mathrm{tr}\rho = 1$ ✓

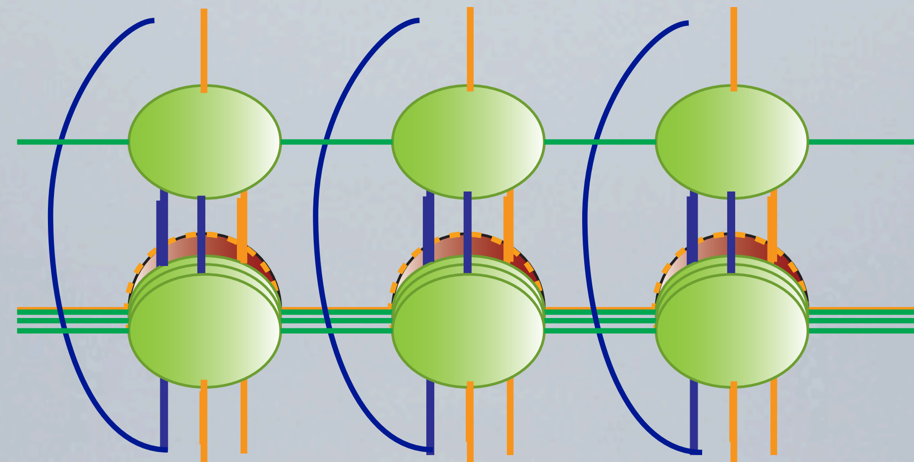
$\rho \geq 0$ ✗



MIXED STATES

- MPDO = Matrix Product Density Operator

purification



need some properties

$$\rho = \sum_{i_1, j_1 \ldots i_N, j_N} \mathrm{tr}(M_1^{i_1 j_1} M_2^{i_2 j_2} \ldots M_N^{i_N j_N}) |i_1 \ldots i_N\rangle\langle j_1 \ldots j_N|$$

can we impose them *locally*?

$\rho = \rho^\dagger$ ✔

$\mathrm{tr}\rho = 1$ ✔

$\rho \geq 0$ ✘ in a way

$$\rho_S = \mathrm{tr}_A |\Psi_{SA}\rangle\langle\Psi_{SA}|$$

see Werner et al., arXiv:1412.5746

MIXED STATES

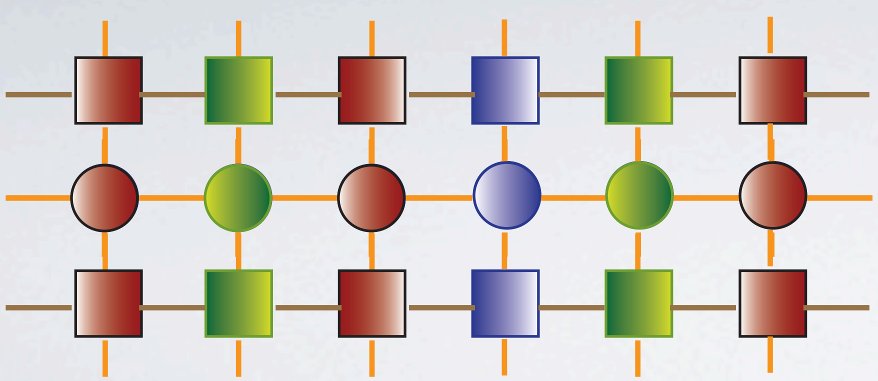
• MPO = Matrix Product Operator

Similar problems can be attacked

equilibrium → thermal states

imaginary time evolution

time-dependent → real time evolution



unitary $\rho(t) = U(t)\rho(0)U(t)^\dagger$

non-unitary $$\frac{d\rho(t)}{dt} = \mathcal{L}(\rho)$$

Verstraete et al., PRL 2004

Prosen, Znidaric et al., PRL 2008,...

MIXED STATES

- MPO = Matrix Product Operator

Another possibility for open dynamics

Dissipative real-time dynamics
produces a steady state

$$\frac{d\rho(t)}{dt} = \mathcal{L}(\rho) \longrightarrow \mathcal{L}(\rho_S) = 0$$

fixed point of
Liouvillian map

Approximate it as a MPO

works by García-Ripoll et al., Zwolak, Prosen, ...

MIXED STATES

- MPO = Matrix Product Operator

Methods to find steady state simulating long time evolution

analogous to imaginary time evolution

Here: variational method for steady state MPO

analogous to DMRG

works by García-Ripoll et al., Zwolak, Prosen, ...

VARIATIONAL STEADY STATES

Applications:

interesting kind of models

dissipative QPT | dissipative QC

collective effects | entangled states

Potential advantages of variational strategy:

faster convergence than time evolution

real time evolution needs to be followed

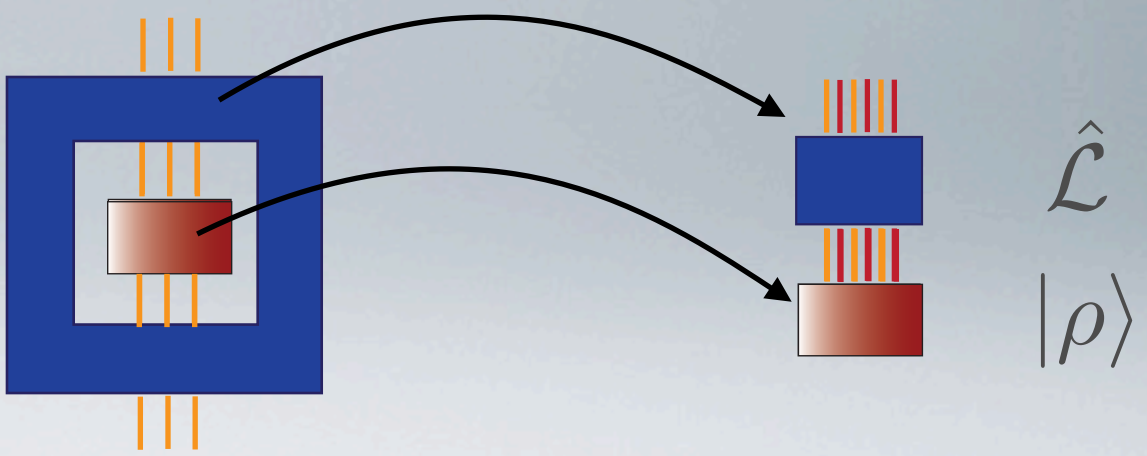
(operator) entanglement in the MPO?

VARIATIONAL STEADY STATES

METHOD

Dynamics determined by Liouvillian

$$\frac{d\rho}{dt} = \mathcal{L}(\rho)$$



vectorize $|\rho\rangle$

superoperator $\hat{\mathcal{L}}$

WANTED

fixed point of evolution

Search for the null vector

$$\hat{\mathcal{L}}|\rho\rangle = 0$$

see also Mascarenhas et al., PRA92, 022116 (2015)

VARIATIONAL STEADY STATES

METHOD

Analogy to GS search

Hermitian	H	$\hat{\mathcal{L}}$	non-Hermitian
	$\min\ \lambda$	$\lambda = 0$	$e^{\hat{\mathcal{L}}}\lvert\rho_S\rangle = \lvert\rho_S\rangle$
	$\lvert\Psi_{\mathrm{GS}}\rangle$	$\lvert\rho_{\mathrm{S}}\rangle$	

$$\hat{\mathcal{L}}^\dagger \hat{\mathcal{L}} |\rho_S\rangle = 0$$

lowest eigenvalue

VARIATIONAL STEADY STATES

METHOD

Master equation of Lindblad form

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\rho L_k^\dagger L_k - \frac{1}{2} L_k^\dagger L_k \rho \right)$$

~*local* (MPO) *local*

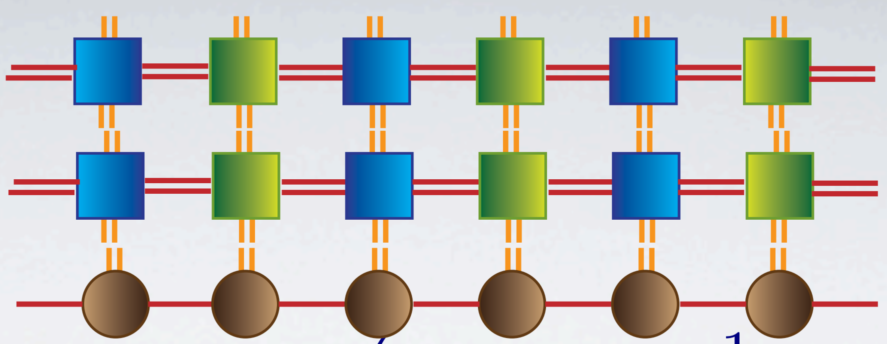
$\hat{\mathcal{L}}$ MPO ⟶ $\hat{\mathcal{L}}^\dagger \hat{\mathcal{L}}$

$$\frac{d|\rho\rangle}{dt} = \left[-i(H\otimes I - I \otimes H^T) + \sum_k \gamma_k \left(L_k \otimes L_k^* - \frac{1}{2} I \otimes L_k^T L_k^* - \frac{1}{2} L_k^\dagger L_k \otimes I \right) \right] |\rho\rangle$$

VARIATIONAL STEADY STATES

METHOD

Master equation of Lindblad form



$$\frac{d|\rho\rangle}{dt} = \left[-i(H \otimes I - I \otimes H^T) + \sum_k \gamma_k \left(L_k \otimes L_k^* - \frac{1}{2} I \otimes L_k^T L_k^* - \frac{1}{2} L_k^\dagger L_k \otimes I \right) \right] |\rho\rangle$$

lowest eigenvalue

VARIATIONAL STEADY STATES

POTENTIAL ISSUES

Positivity

fixed point of the evolution

Accuracy of MPO approximation

Degeneracies

maybe smaller gaps? ⇒ metastable states?

local effective Lindblad operator does not preserve any property ⇒ symmetries?

Prosen, Znidaric, 2009

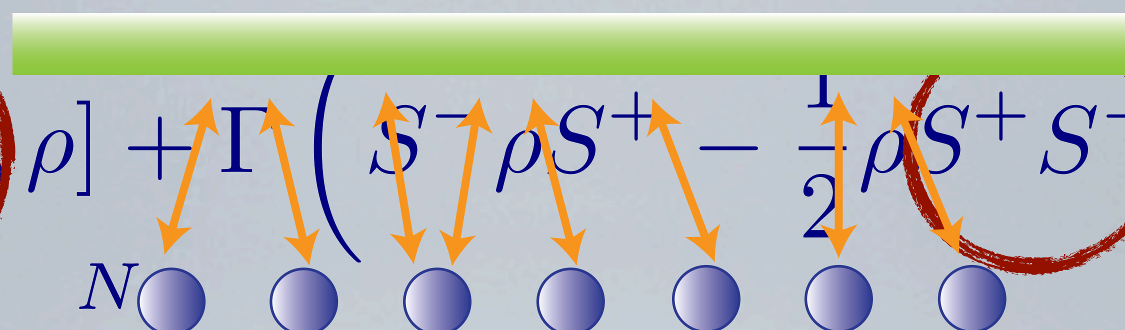
Kastoryano, Eisert, 2013

NUMERICAL RESULTS

DICKE MODEL

N 2-level atoms coupled to same EM mode

$$\frac{d\rho}{dt} = -i\Omega[S_x, \rho] + \Gamma\left(S^- \rho S^+ - \frac{1}{2}\rho S^+ S^- - \frac{1}{2} S^+ S^- \rho\right)$$



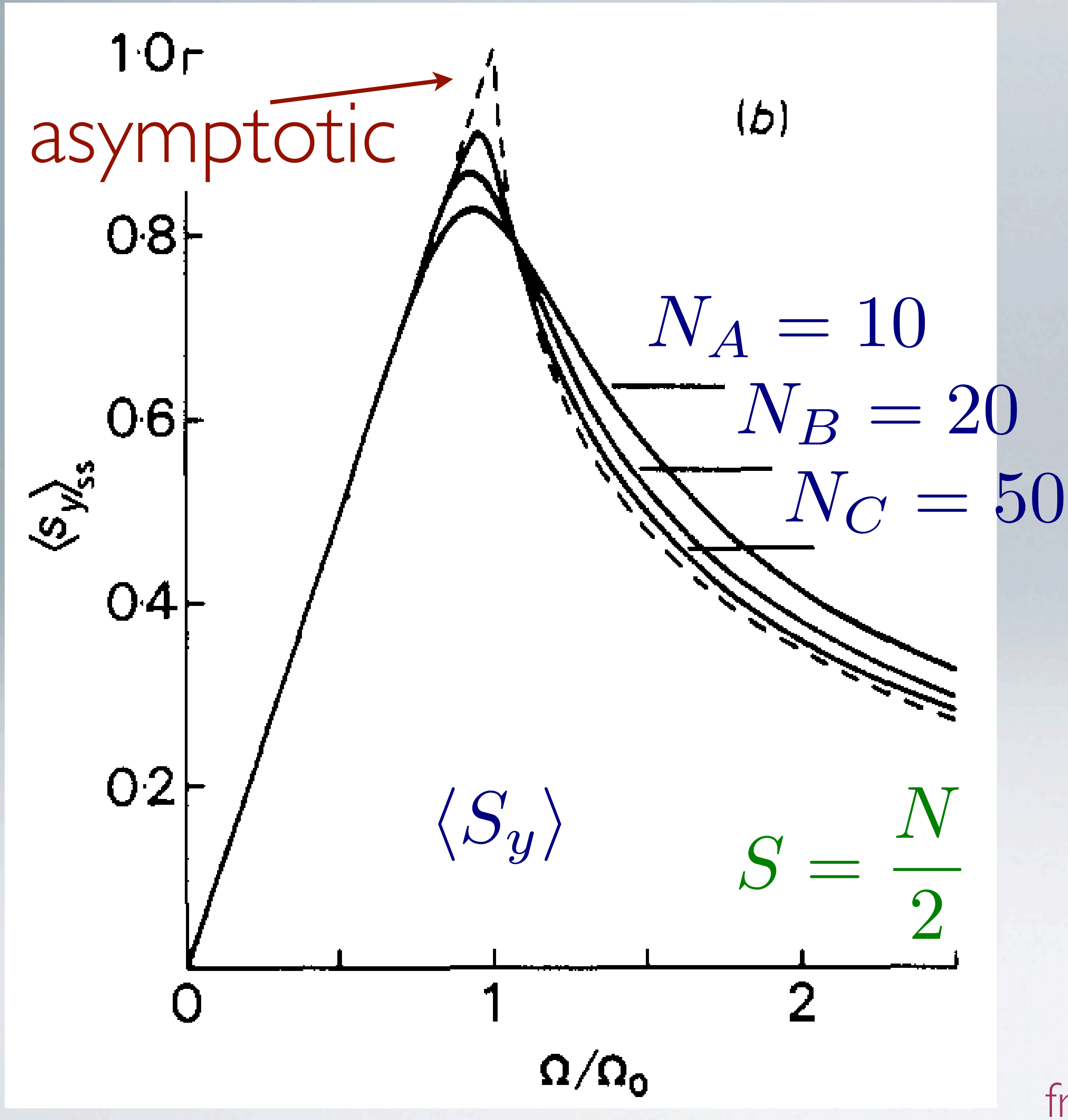
$$S_x = \sum_{n=1}^{N} s_x$$ collective coupling

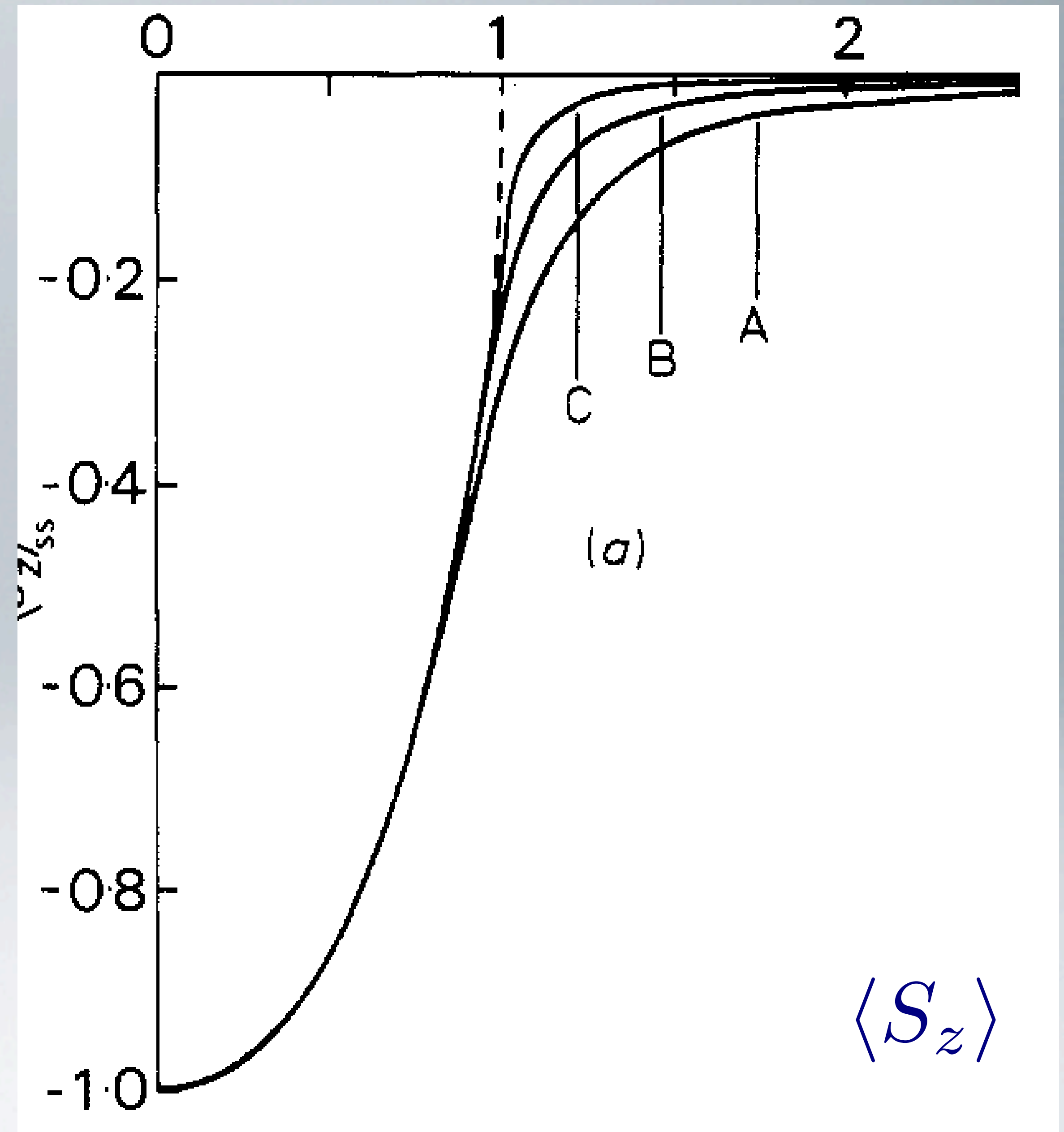
phase transition to superradiant phase $\frac{\Omega}{\Gamma} = \frac{N}{2}$

analytic solution conserved total spin

Dicke, 1954
Hepp, Lieb, 1973
Carmichael, 1980

DICKE MODEL





from Carmichael, J Phys B 1980

DICKE MODEL

is an interesting model...

phase transitions dissipative

collective phenomena

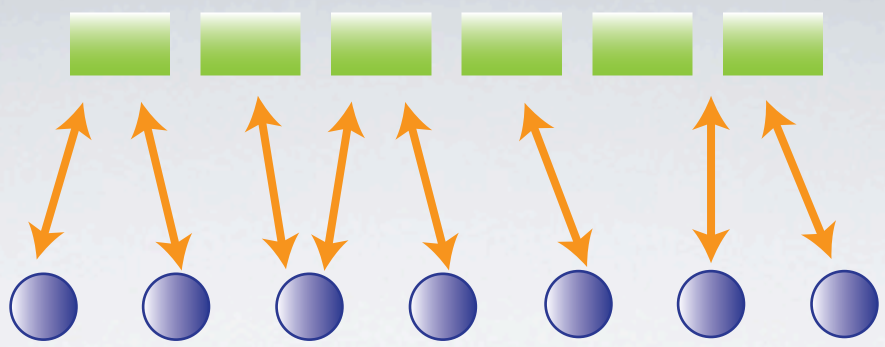
entanglement

but experimentally difficult

Baumann et al., 2010
Hamner et al., 2014
Baden et al., 2014

Do *simpler* models show similar phenomena?

more local



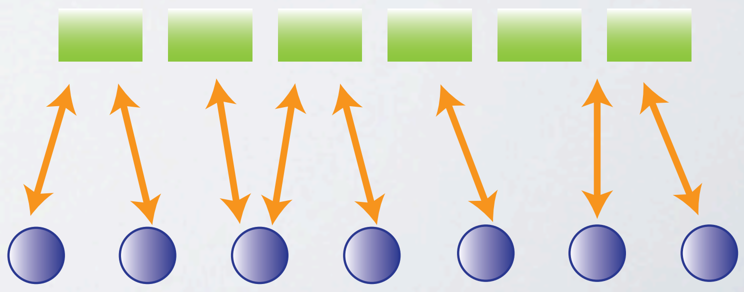
A SIMPLER MODEL

Lower dimensional version of Dicke model

N 2-level systems with dissipation coupling NN

$$\frac{d\rho}{dt} = -i\Omega[S_x, \rho] + \Gamma \sum_n \left(S^-_{n\ n+1} \rho S^+_{n\ n+1} - \frac{1}{2} \rho S^+_{n\ n+1} S^-_{n\ n+1} - \frac{1}{2} S^+_{n\ n+1} S^-_{n\ n+1} \rho \right)$$

$$S^+_{n\ n+1} = \sigma^+_{n+1} \otimes I + I \otimes \sigma^+_{n+1}$$



LOW DIM DICKE MODEL

Liouvillian can be expressed as a *small* MPO of

$$\chi = 5$$

Approximate the steady state by a MPO

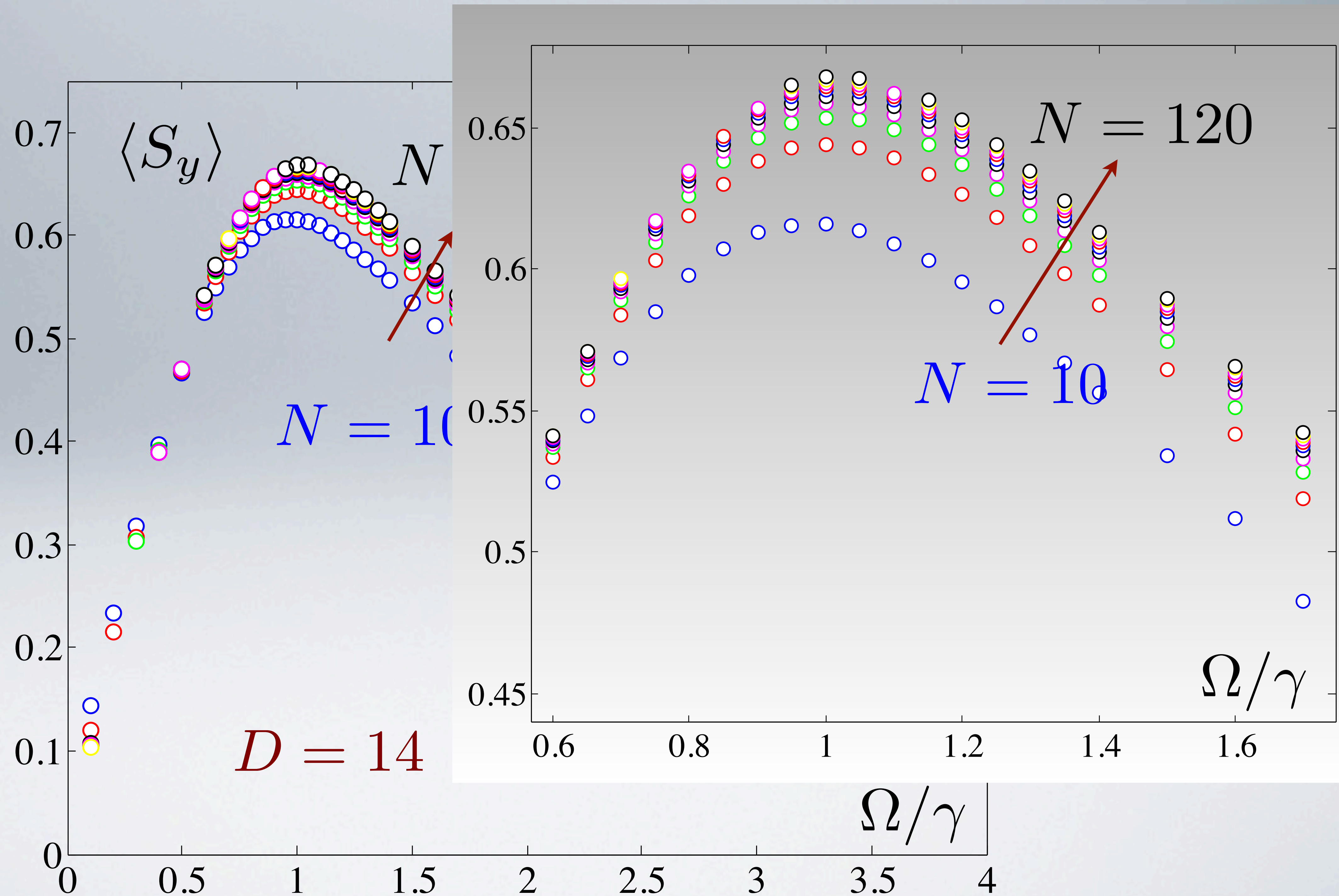
fixed D → check convergence

no explicit positivity

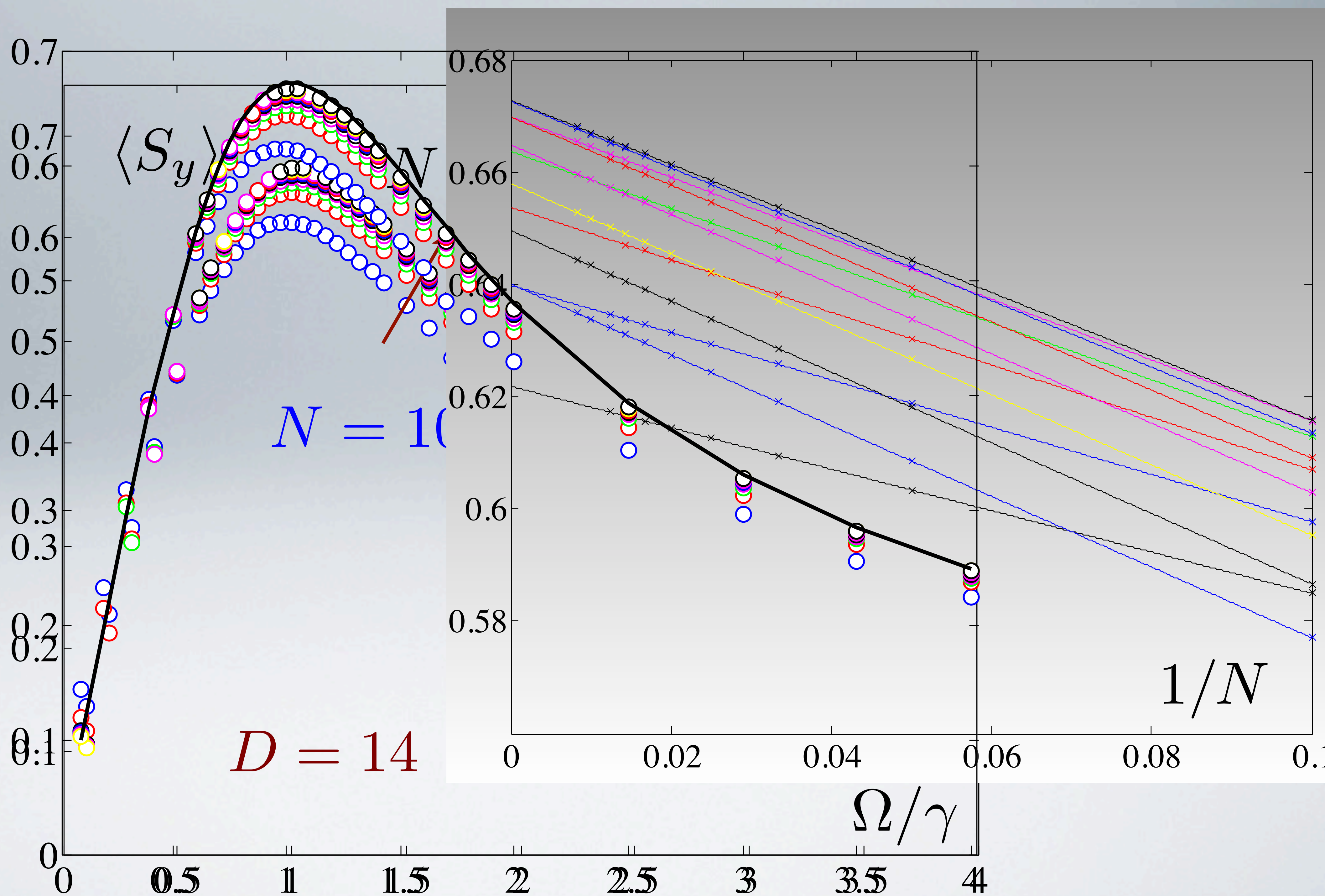
no explicit Hermiticity

No special symmetry

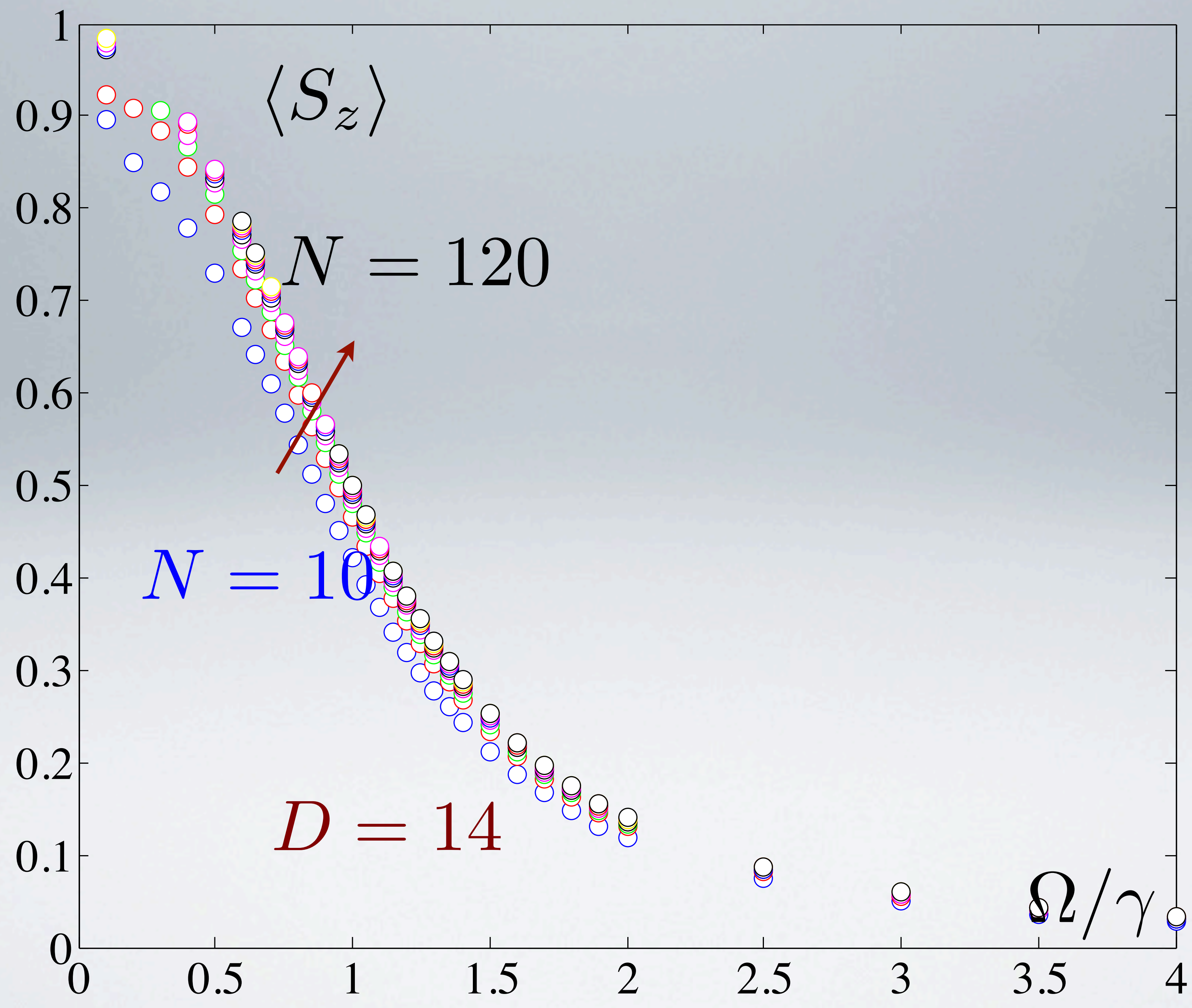
LOW DIM DICKE MODEL



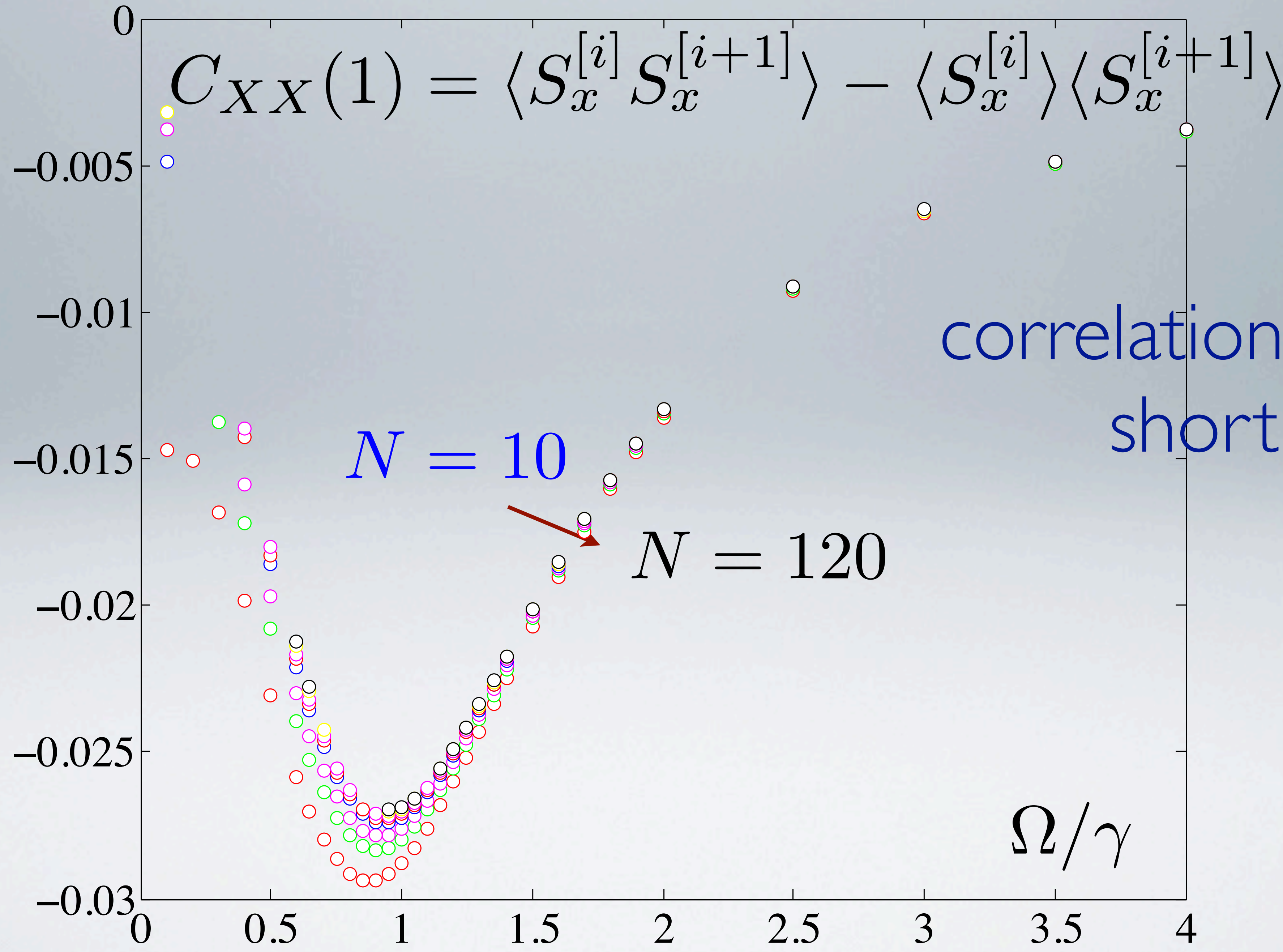
LOW DIM DICKE MODEL



LOW DIM DICKE MODEL

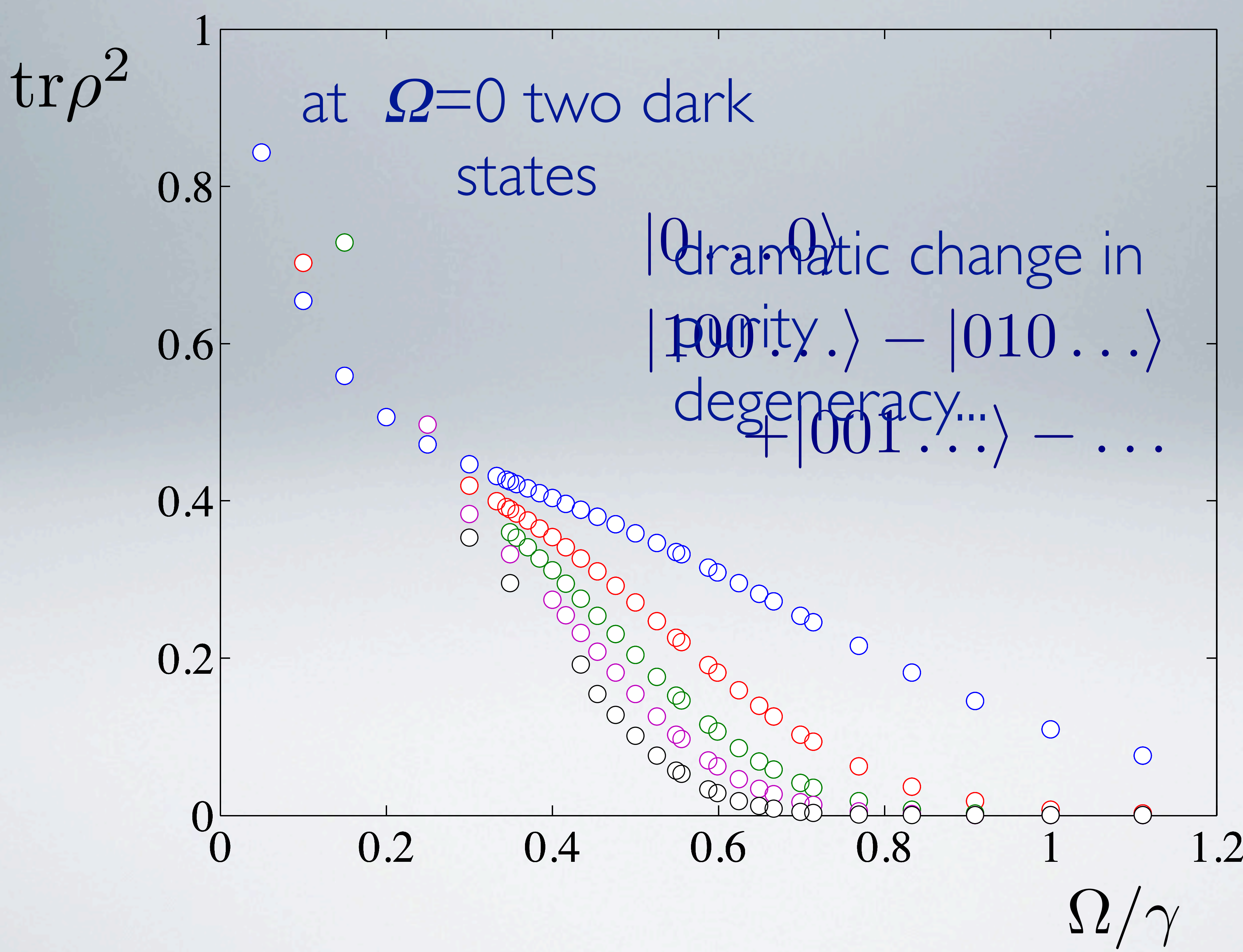


LOW DIM DICKE MODEL



correlations, but short

LOW DIM DICKE MODEL



Other interesting models...

DISSIPATIVE ISING CHAIN

$$H = \sum_n \left[\frac{V}{2} \sigma_z^{[n]} \sigma_z^{[n+1]} + \frac{\Omega}{2} \sigma_x^{[n]} + \frac{\Delta - V}{2} \sigma_z^{[n]} \right]$$

local dissipation

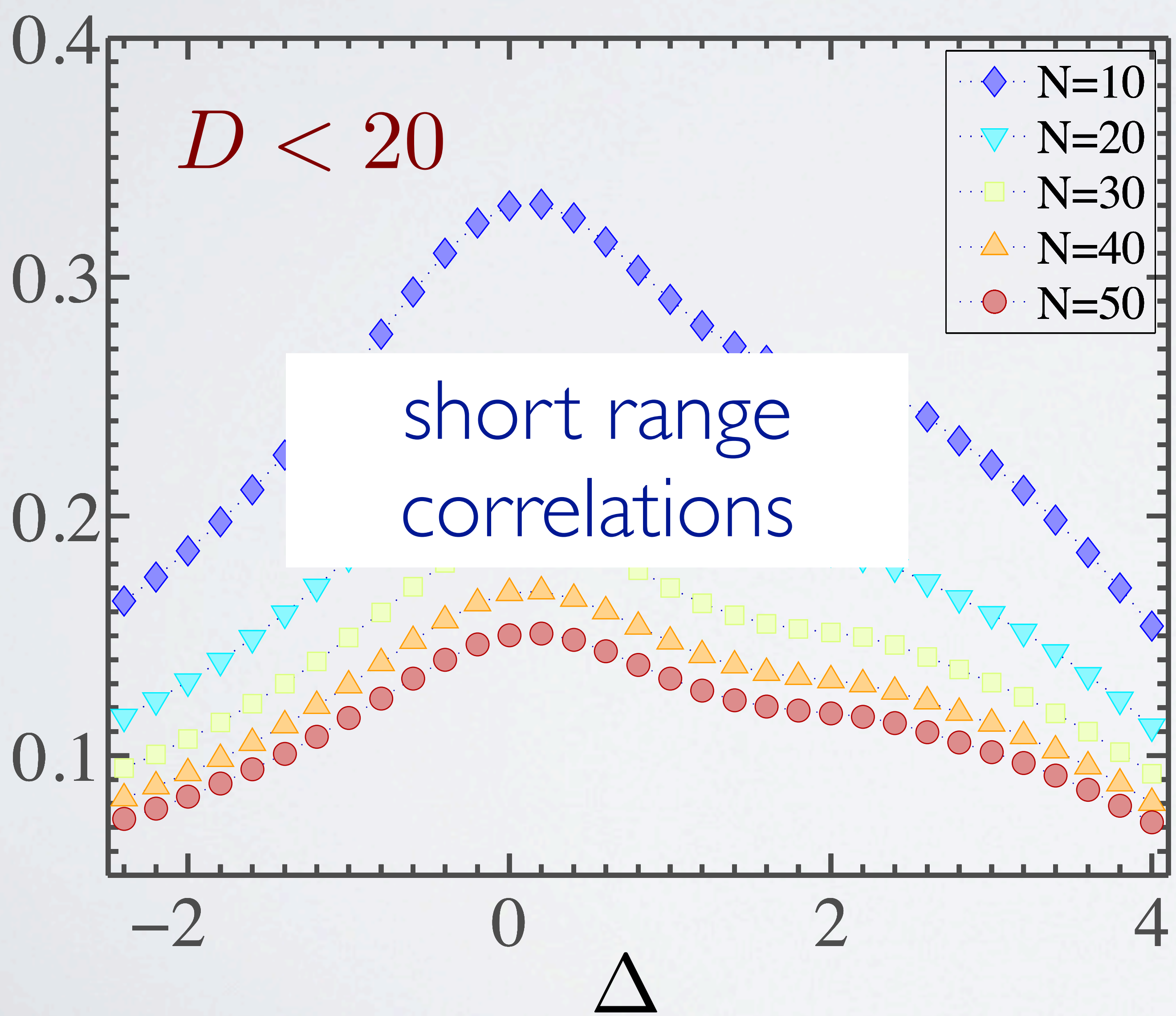
$$L_n = \sqrt{\gamma} \sigma_n^+$$

can be realized by Rydberg atoms

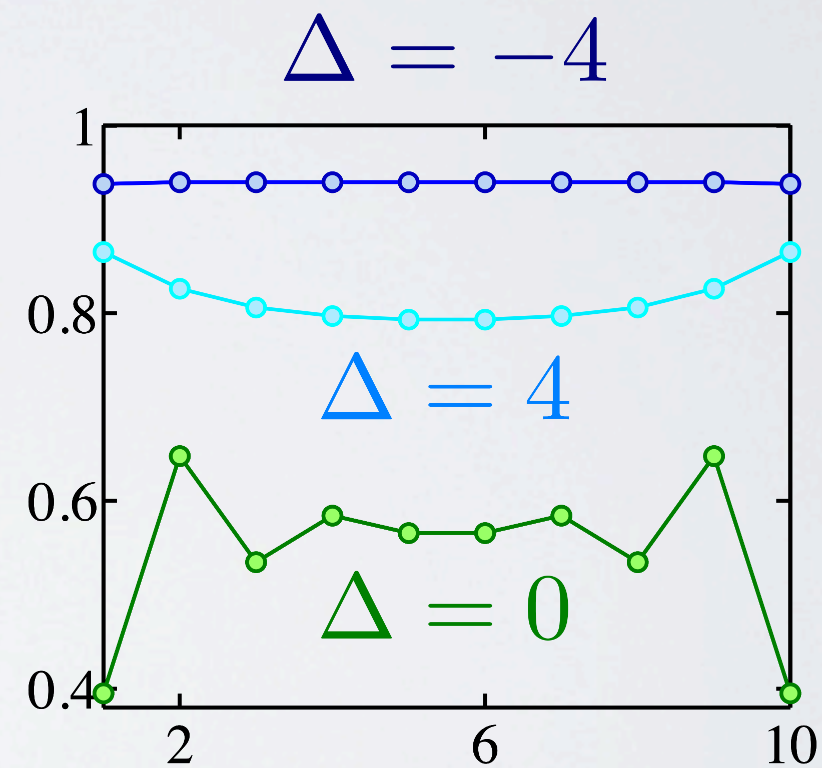
steady state can show AFM ordering

DISSIPATIVE ISING CHAIN

AF order
(staggered magnetization)



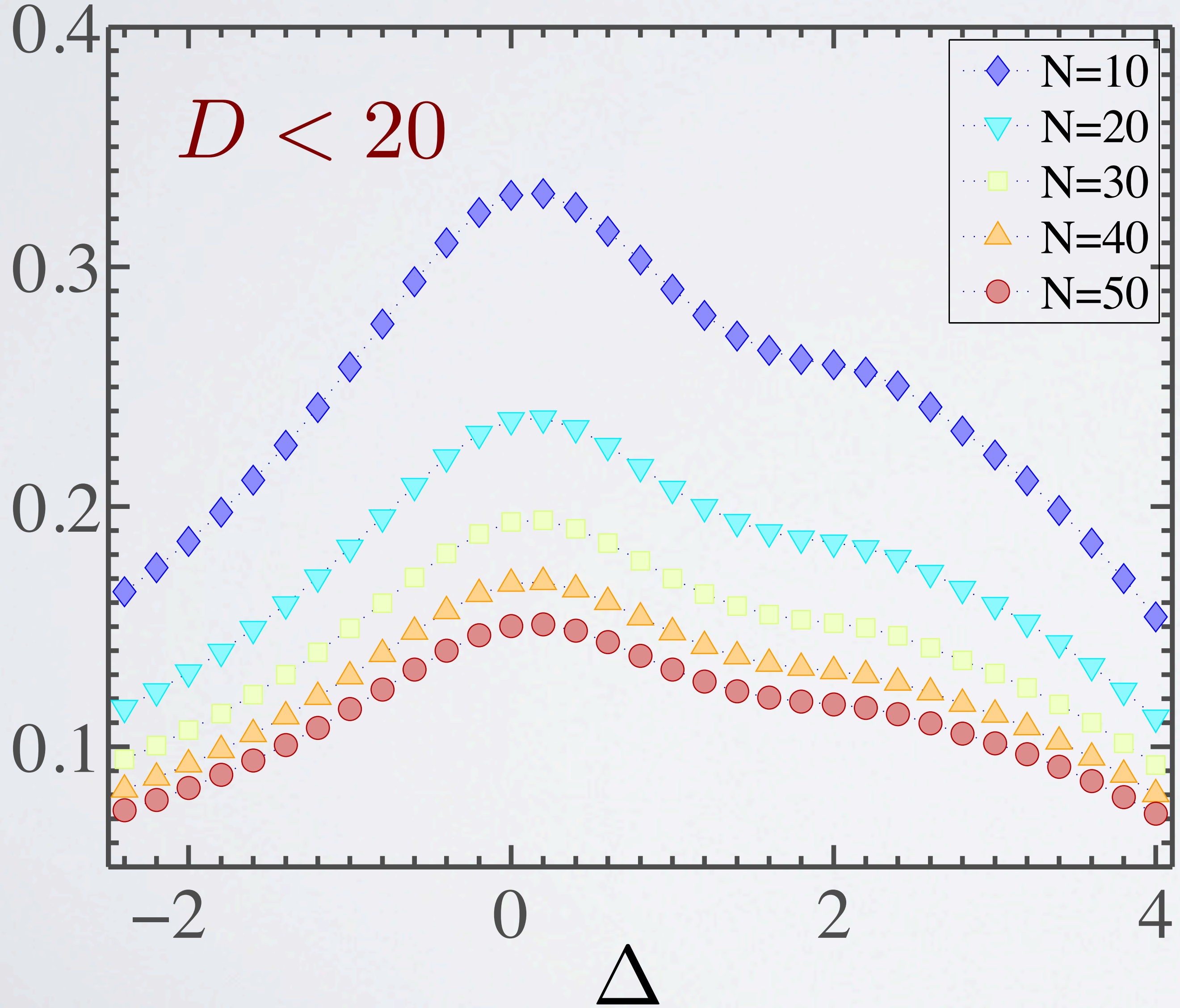
local polarization
$\langle \sigma_z^{[n]} \rangle$



$\gamma = 1,\ V = 5,\ \Omega = 1.5$

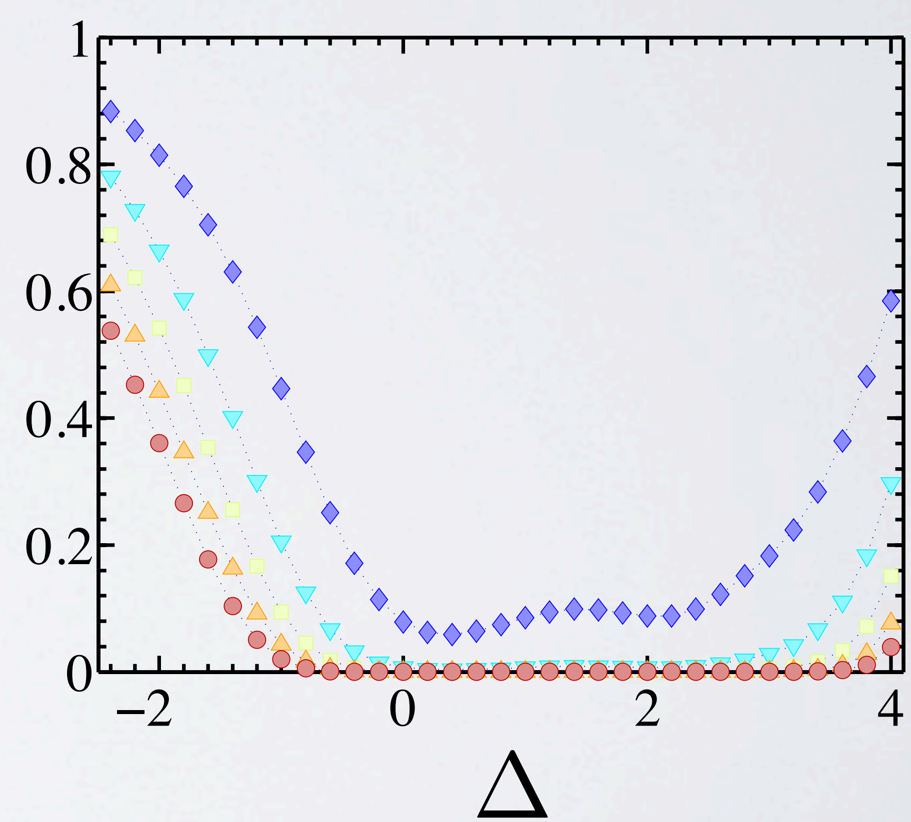
DISSIPATIVE ISING CHAIN

AF order
(staggered magnetization)



$|\Delta| \to \infty$
completely polarized steady state

purity



$\gamma = 1,\ V = 5,\ \Omega = 1.5$

Interesting models...

ISING + COHERENT DISSIPATION

$$H = \sum_n \sigma_z^{[n]}\sigma_z^{[n+1]} + g\sigma_x^{[n]}$$

coherent dissipation

$$L_n = \mu\sigma_n^+ - \nu\sigma_{n+1}^-$$

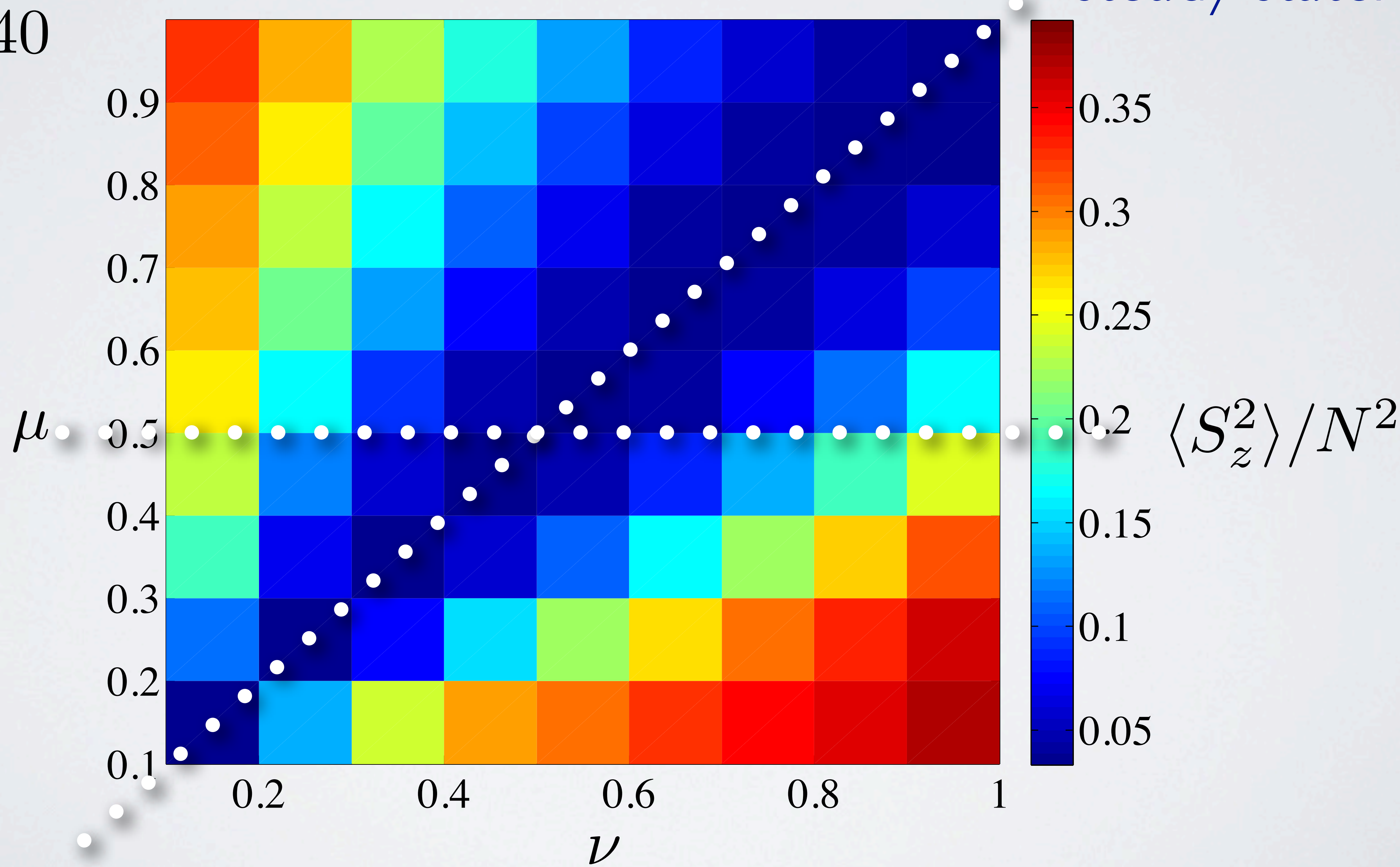
both Hamiltonian and dissipation can induce coherence

ISING + COHERENT DISSIPATION

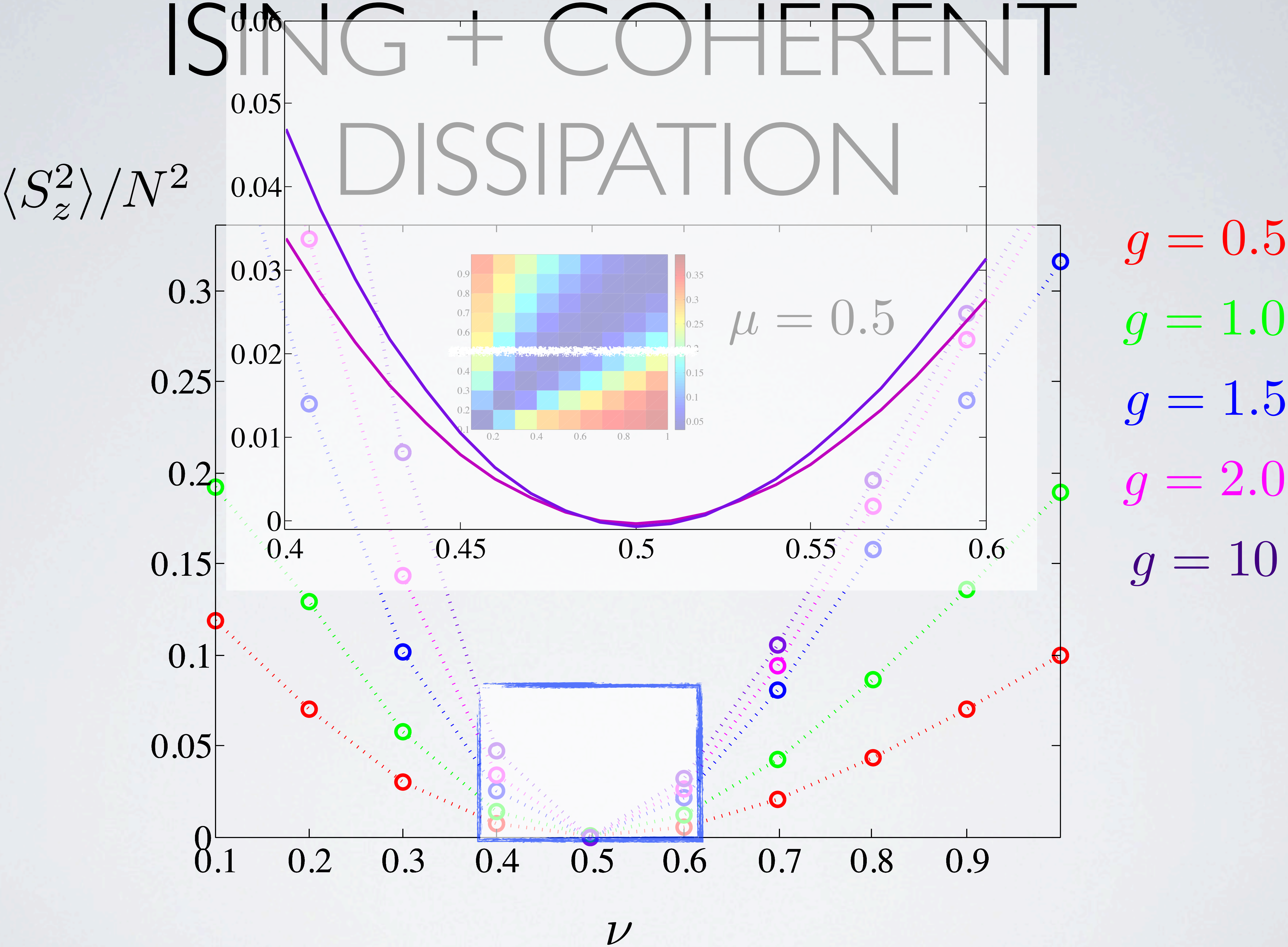
at **μ=ν** the identity is a steady state!

$N = 40$

$g = 1$



ISING + COHERENT DISSIPATION



SUMMARY

NESS can be found variationally

Very good convergence (varying models, parameters)

Very small bond dimension required

Stability can be delicate

Warm-up phase needed!

Future: symmetries, trace one, degeneracies...

much to understand about MPDOs representations

PRL 114, 220601 (2015)

THANKS!

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THANKS!