Matrix product solutions of boundary driven quantum chains

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Program of the talk

- Boundary driven interacting quantum chain paradigm and exact solutions via Matrix Product Ansatz
- New conservation laws and exact bounds on transport coefficients
- Open problems

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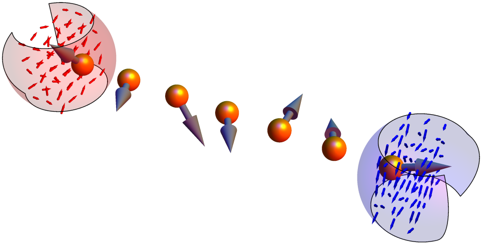
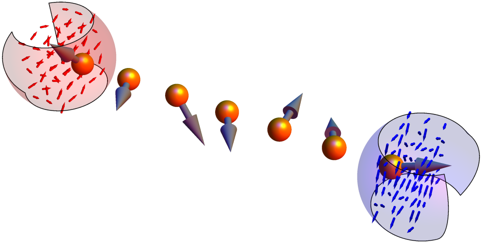


Collaborators involved in this work:
Enej Ilievski (now at University of Amsterdam), Slava Popkov (Cologne/Florence), Marko Medenjak (PhD student, Ljubljana)

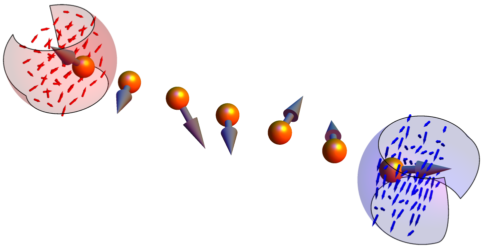
Nonequilibrium quantum transport problem in one-dimension



Canonical markovian master equation for the many-body density matrix:

The Lindblad (L-GKS) equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \hat{\mathcal{L}}\rho := -\mathrm{i}[H,\rho] + \sum_{\mu} \left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\} \right).$$



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- *Bulk*: Fully **coherent**, local interactions,e.g. $H = \sum_{x=1}^{n-1} h_{x,x+1}$.
- *Boundaries:* Fully **incoherent**, ultra-local dissipation, jump operators L_μ supported near boundaries $x = 1$ or $x = n$.

Steady state Lindblad equation $\hat{\mathcal{L}}\rho_\infty = 0$:

$$\mathrm{i}[H, \rho_\infty] = \sum_\mu \left(2L_\mu \rho_\infty L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho_\infty\}\right)$$

The *XXZ* Hamiltonian:

$$H = \sum_{x=1}^{n-1} (2\sigma_x^+ \sigma_{x+1}^- + 2\sigma_x^- \sigma_{x+1}^+ + \Delta \sigma_x^{\mathrm{z}} \sigma_{x+1}^{\mathrm{z}})$$

and symmetric boundary (ultra local) Lindblad jump operators:

$$\begin{aligned} L_1^{\mathrm{L}} &= \sqrt{\frac{1}{2}(1-\mu)\varepsilon}\;\sigma_1^+, \quad L_1^{\mathrm{R}} = \sqrt{\frac{1}{2}(1+\mu)\varepsilon}\;\sigma_n^+, \\ L_2^{\mathrm{L}} &= \sqrt{\frac{1}{2}(1+\mu)\varepsilon}\;\sigma_1^-, \quad L_2^{\mathrm{R}} = \sqrt{\frac{1}{2}(1-\mu)\varepsilon}\;\sigma_n^-. \end{aligned}$$

Two key boundary parameters:

- ε System-bath coupling strength
- μ Non-equilibrium driving strength (bias)

TP, PRL**106**(2011); PRL**107**(2011); Karevski, Popkov, Schütz, PRL**111**(2013)

$$\rho_\infty = (\operatorname{tr} R)^{-1} R, \quad R = \Omega\Omega^\dagger$$

$$\Omega = \sum_{(s_1,\ldots,s_n)\in\{+,-,0\}^n} \langle 0|\mathbf{A}_{s_1}\mathbf{A}_{s_2}\cdots\mathbf{A}_{s_n}|0\rangle \sigma^{s_1}\otimes\sigma^{s_2}\cdots\otimes\sigma^{s_n} = \langle 0| \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_+ \\ \mathbf{A}_- & \mathbf{A}_0 \end{pmatrix}^{\otimes n} |0\rangle$$

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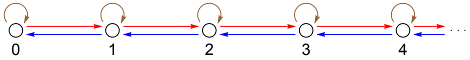
$$\begin{aligned}
\mathbf{A}_0 &= \sum_{k=0}^{\infty} a_k^0 |k\rangle\langle k|, \\
\mathbf{A}_+ &= \sum_{k=0}^{\infty} a_k^+ |k\rangle\langle k+1|, \\
\mathbf{A}_- &= \sum_{k=0}^{\infty} a_k^- |k+1\rangle\langle r|,
\end{aligned}$$



Cholesky decomposition of NESS and Matrix Product Ansatz (for $\mu = 1$)

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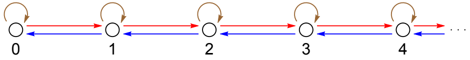
$$\mathbf{A}_0 = \sum_{k=0}^{\infty} a^0_k |k\rangle\langle k|,$$

$$\mathbf{A}_+ = \sum_{k=0}^{\infty} a^+_k |k\rangle\langle k+1|,$$



$$\mathbf{A}_- = \sum_{k=0}^{\infty} a^-_k |k+1\rangle\langle r|,$$

$$a^0_k = \cos((s-k)\eta) \qquad \cos\eta := \Delta,$$

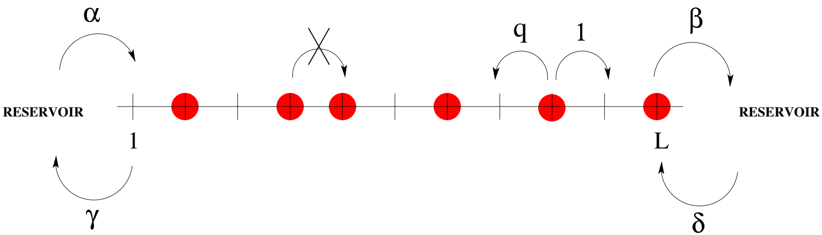
$$a^+_k = \sin((k+1)\eta) \qquad \tan(\eta s) := \frac{\varepsilon}{2\mathrm{i}\sin\eta}$$

$$a^-_k = \cos((2s-k)\eta)$$

s is a q$-$deformed complex spin $q = e^{\mathrm{i}\eta}$

Markovian model on a 2^L dimensional probability state vector $\underline{p}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{p} = M\underline{p}$$



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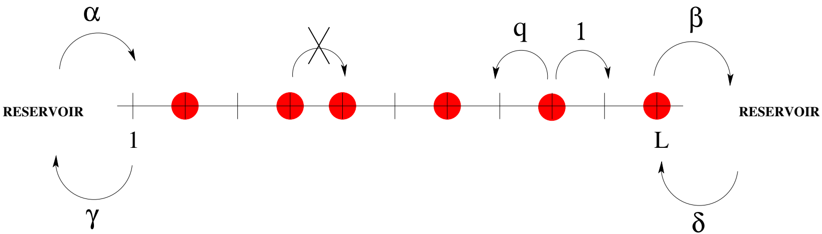

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Nonequilibrium steady state (NESS): a fixed point probability state vector $\underline{p}_\infty$

$$M\underline{p}_\infty = 0$$

Matrix Product Ansatz (MPA)

Derrida, Evans, Hakim & Pasquier (1993):

Let $\mathbf{A}_0, \mathbf{A}_1$ be a pair of matrices, and $\langle L|$, $|R\rangle$ a pair of left and right 'vacua'.

$$\mathrm{MPA}: \qquad p_{s_1,s_2,\ldots,s_L} = \langle L|\mathbf{A}_{s_1}\mathbf{A}_{s_2}\cdots\mathbf{A}_{s_L}|R\rangle, \quad s_j \in \{0,1\}$$

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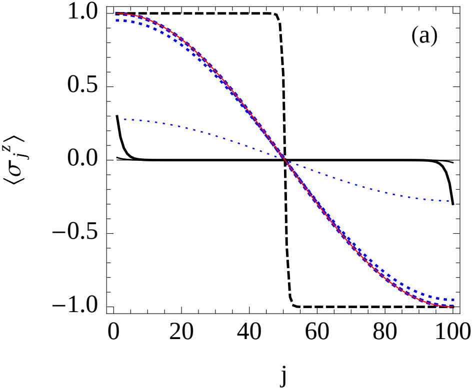
Asking such MPA $\underline{p}$ to solve the Markov fixed point condition $M\underline{p} = 0$ results in a single algebraic relation in the bulk

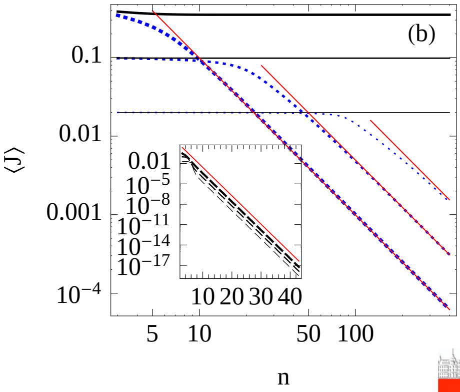
$$\mathbf{A}_1\mathbf{A}_0 - q\mathbf{A}_0\mathbf{A}_1 = (1-q)(\mathbf{A}_0 + \mathbf{A}_1)$$

with two boundary conditions

$$\langle L|(\alpha\mathbf{A}_0 - \gamma\mathbf{A}_1) = \langle L|, \quad (\beta\mathbf{A}_1 - \delta\mathbf{A}_0)|R\rangle = |R\rangle$$

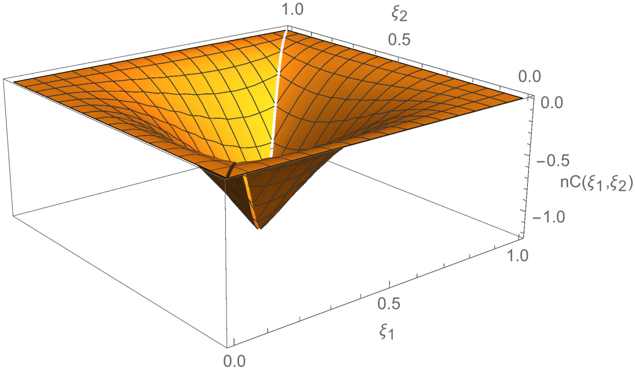
- For $|\Delta| < 1$, $\langle J \rangle \sim n^0$ (ballistic)
- For $|\Delta| > 1$, $\langle J \rangle \sim \exp(-\mathrm{const}\, n)$ (insulating)
- For $|\Delta| = 1$, $\langle J \rangle \sim n^{-2}$ (anomalous)





$$C\left(\frac{x}{n}, \frac{y}{n}\right) = \langle \sigma^z_x \sigma^z_y \rangle - \langle \sigma^z_x \rangle \langle \sigma^z_y \rangle$$

for isotropic case $\Delta = 1$ (XXX)



$$C(\xi_1, \xi_2) = -\frac{\pi^2}{2n} \xi_1 (1 - \xi_2) \sin(\pi \xi_1) \sin(\pi \xi_2), \quad \text{for} \quad \xi_1 < \xi_2$$

TP, Ilievski and Popkov, NJP(2013)

Cholesky-factor (amplitude operator) $\Omega = \Omega_n(\varepsilon)$ satisfies the "square-root" Lindblad equation:

$$[H, \Omega_n(\varepsilon)] = -\mathrm{i}\varepsilon\sigma^{\mathrm{z}} \otimes \Omega_{n-1}(\varepsilon) + \mathrm{i}\varepsilon\Omega_{n-1}(\varepsilon) \otimes \sigma^{\mathrm{z}}.$$

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This follows after considering the Lax operator $\mathbf{L} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathcal{H}_a)$

$$\mathbf{L}(\varphi, s) = \begin{pmatrix} \sin(\varphi + \eta \mathbf{S}_s^{\mathrm{z}}) & (\sin\eta)\mathbf{S}_s^{-} \\ (\sin\eta)\mathbf{S}_s^{+} & \sin(\varphi - \eta \mathbf{S}_s^{\mathrm{z}}) \end{pmatrix}$$

where $\mathbf{S}_s^{\pm,\mathrm{z}}$ is the *highest-weight* complex-spin irep of $U_q(\mathfrak{sl}_2)$ over $\mathcal{H}_{\mathrm{a}}$:

$$\begin{aligned} \mathbf{S}_s^{\mathrm{z}} &= \sum_{k=0}^{\infty} (s-k)|k\rangle\langle k|, \\ \mathbf{S}_s^{+} &= \sum_{k=0}^{\infty} \frac{\sin(k+1)\eta}{\sin\eta} |k\rangle\langle k+1|, \\ \mathbf{S}_s^{-} &= \sum_{k=0}^{\infty} \frac{\sin(2s-k)\eta}{\sin\eta} |k+1\rangle\langle k|. \end{aligned}$$

and writing

$$\Omega_n(\varepsilon) = \langle 0|_{\mathrm{a}} \mathbf{L}_{1,a}\mathbf{L}_{2,a}\cdots\mathbf{L}_{n,a}|0\rangle_{\mathrm{a}}, \quad \text{with} \quad \varphi = \frac{\pi}{2}, \quad \tan(\eta s) := \frac{\varepsilon}{2\mathrm{i}\sin\eta}.$$

Steady state Lindblad eq. in fact follows from telescoping series using the **local operator divergence** condition (or Sutherland equation)

$$[h_{x,x+1}, \mathbf{L}_{x,\mathrm{a}} \mathbf{L}_{x+1,\mathrm{a}}] = \tilde{\mathbf{L}}_{x,\mathrm{a}}\, \mathbf{L}_{x+1,\mathrm{a}} - \mathbf{L}_{x,\mathrm{a}}\, \tilde{\mathbf{L}}_{x+1,\mathrm{a}}, \quad \tilde{\mathbf{L}}_{x,\mathrm{a}}(\varphi, s) \equiv \partial_\varphi \mathbf{L}_{x,\mathrm{a}}(\varphi, s).$$

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This, in turn, is equivalent to YBE (or so-called RLL relation)

$$\check{R}_{1,2}(\varphi_2-\varphi_1)\mathbf{L}_{1,\mathrm{a}}(\varphi_1,s)\mathbf{L}_{2,\mathrm{a}}(\varphi_2,s) = \mathbf{L}_{1,\mathrm{a}}(\varphi_2,s)\mathbf{L}_{2,\mathrm{a}}(\varphi_1,s)\check{R}_{1,2}(\varphi_2-\varphi_1)$$

where $R_{1,2} = \check{R}_{1,2}P_{1,2}$ is the 6-vertex R-matrix yielding the XXZ hamiltonian as

$$h_{1,2} = 2\partial_\varphi \check{R}_{1,2}|_{\varphi=0}.$$

However, using the concept of **universal R-matrix** there is an infinitely dimensional matrix interwining the *representation parameters*

$$\check{\mathbf{R}}_{a_1,a_2}\mathbf{L}_{x,a_1}(\varphi_1,s_1)\mathbf{L}_{x,a_2}(\varphi_2,s_2)=\mathbf{L}_{x,a_1}(\varphi_2,s_2)\mathbf{L}_{x,a_2}(\varphi_1,s_1)\check{\mathbf{R}}_{a_1,a_2}$$

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Remarkably, together with the property

$$\langle 0|_{\mathrm{a}_1}\langle 0|_{\mathrm{a}_2}\check{\mathbf{R}}_{\mathrm{a}_1,\mathrm{a}_2}=\langle 0|_{\mathrm{a}_1}\langle 0|_{\mathrm{a}_2},\quad \check{\mathbf{R}}_{\mathrm{a}_1,\mathrm{a}_2}|0\rangle_{\mathrm{a}_1}|0\rangle_{\mathrm{a}_2}=|0\rangle_{\mathrm{a}_1}|0\rangle_{\mathrm{a}_2}$$

this immediately implies the commutatvity $[W_n(\varphi_1,s_1),W_n(\varphi_2,s_2)]=0$ of the generalised highest-weight transfer matrix

$$W_n(\varphi,s)=\langle 0|_{\mathrm{a}}\mathbf{L}_{1,\mathrm{a}}(\varphi,s)\cdots\mathbf{L}_{n,\mathrm{a}}(\varphi,s)|0\rangle_{\mathrm{a}},\quad \Omega_n(\varepsilon)=W_n(\pi/2,s(\varepsilon)).$$

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Namely,

$$\begin{aligned}
W_n(\varphi_1,s_1)W_n(\varphi_2,s_2) &= \langle 0|_{\mathrm{a}_1}\langle 0|_{\mathrm{a}_2}\check{\mathbf{R}}_{\mathrm{a}_1,\mathrm{a}_2}\prod_{x=1}^{n}\mathbf{L}_{x,\mathrm{a}_1}(\varphi_1,s_1)\mathbf{L}_{x,\mathrm{a}_2}(\varphi_2,s_2)|0\rangle_{\mathrm{a}_1}|0\rangle_{\mathrm{a}_2}\\
&= \langle 0|_{\mathrm{a}_1}\langle 0|_{\mathrm{a}_2}\prod_{x=1}^{n}\mathbf{L}_{x,\mathrm{a}_1}(\varphi_2,s_2)\mathbf{L}_{x,\mathrm{a}_2}(\varphi_1,s_1)\check{\mathbf{R}}_{\mathrm{a}_1,\mathrm{a}_2}|0\rangle_{\mathrm{a}_1}|0\rangle_{\mathrm{a}_2}\\
&= W_n(\varphi_2,s_2)W_n(\varphi_1,s_1)
\end{aligned}$$

Local conserved operators $[H, Q_m] = [Q_m, Q_l] = 0$, typically derived via Algebraic Bethe Ansatz machinery

$$Q_m = \partial_\varphi^m \log \operatorname{tr}_{\mathrm{a}} \mathbf{L}^{\otimes_x n}(\varphi, \tfrac{1}{2})|_{\varphi = \frac{\eta}{2}} = \sum_x q_x^{(m)}, \quad Q_1 \propto H$$

satisfy extensivity property ($\langle \bullet \rangle = \operatorname{tr}(\bullet)/\operatorname{tr} \mathbb{1}$):

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Definition (quasi-locality):

Nonlocal operator $A \in \mathrm{End}((\mathbb{C}^2)^{\otimes n})$, with $n-$independent $\langle (a_k \otimes \mathbb{1}_{n-k}) A \rangle$ for any locally supported fixed a_k and with extensivity property

$$\langle A^\dagger A \rangle \propto n$$

is called quasi-local.

Observations:

(TP, PRL**106**(2011); TP and Ilievski, PRL**111**(2013); TP, NPB**886**(2014); Pereira *et al.*, JSTAT**2014**)

1. All local Q_m are spin-flip invariant

$$PQ_m = Q_m P, \quad P = (\sigma^{\rm x})^{\otimes n}.$$

since $PW_n(\varphi, \frac{1}{2})P^{-1} = W_n(\varphi, \frac{1}{2})$

2. However, for generic $s \in \mathbb{C}$

$$PW_n(\varphi, s)P^{-1} = W_n(\pi - \varphi, s)^T \neq W_n(\varphi, s).$$

3. Derivative w.r.t. s at the scalar point $s = 0$ is *quasi-local* if $\eta = \pi l/m$

$$Z_n(\varphi) = c_n\, \partial_s W_n(\varphi, s)|_{s=0}, \quad \langle Z_n^\dagger(\varphi) Z_n(\varphi)\rangle \propto n \quad \text{for} \quad |\mathrm{Re}\varphi - \frac{\pi}{2}| < \frac{\pi}{2m}$$

and *almost conserved*, e.g., for $\varphi = \frac{\pi}{2}$

$$[H, Z_n] = \sigma_1^{\rm z} - \sigma_n^{\rm z}.$$

4. $Q = \mathrm{i}(Z - Z^\dagger)$ has been the first known quasi-local CL of odd parity

$$PQ = -QP$$

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3. Derivative w.r.t. s at the scalar point $s = 0$ is *quasi-local* if $\eta = \pi l/m$

$$Z_n(\varphi) = c_n \partial_s W_n(\varphi, s)|_{s=0}, \quad \langle Z_n^\dagger(\varphi) Z_n(\varphi)\rangle \propto n \quad \text{for} \quad |\mathrm{Re}\varphi - \frac{\pi}{2}| < \frac{\pi}{2m}$$

   and *almost conserved*, e.g., for $\varphi = \frac{\pi}{2}$

$$[H, Z_n] = \sigma_1^{\mathrm{z}} - \sigma_n^{\mathrm{z}}.$$

4. $Q = \mathrm{i}(Z - Z^\dagger)$ has been the first known quasi-local CL of odd parity

$$PQ = -QP$$

Green-Kubo formulae express the conductivities in terms of current a.c.f.

$$\kappa(\omega) = \lim_{t\to\infty} \lim_{n\to\infty} \frac{\beta}{n} \int_0^t \mathrm{d}t' e^{\mathrm{i}\omega t} \langle J(t') J(0) \rangle_\beta$$

When d.c. conductivity diverges, one defines a Drude weight D

$$\kappa(\omega) = 2\pi D \delta(\omega) + \kappa_{\mathrm{reg}}(\omega)$$

which in linear response expresses as

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For integrable quantum systems, Zotos, Naef and Prelovšek (1997) suggested to use Mazur/Suzuki (1969/1971) bound, estimating Drude weight in terms of local conserved operators Q_j, $[H, Q_j] = 0$:

$$D \geq \lim_{n\to\infty} \frac{\beta}{2n} \sum_m \frac{\langle J Q_m \rangle_\beta^2}{\langle Q_m^2 \rangle_\beta}$$

where operators Q_m are chosen mutually orthogonal $\langle Q_m Q_k \rangle_\beta = 0$ for $m \neq k$.

Implication for linear response spin transport

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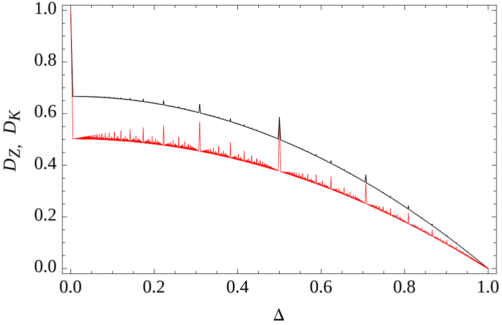
Considering the *spin current* $J = \mathrm{i}\sum_x(\sigma_x^+\sigma_{x+1}^- - \sigma_x^-\sigma_{x+1}^+)$, being of *odd* parity $PJ = -JP$, one has $\langle JQ_j\rangle \equiv 0$, so Mazur bound is trivial.

Mazur bound with the novel quasi-local CL

However, almost conserved odd quasi-local operator Q, or correspondingly extended holomorphic family $K = \{Q(\varphi)\}$, can be used to bound the spin Drude weight:

Fractal Drude weight bound

$$\frac{D}{\beta} \geq D_Z := \frac{\sin^2(\pi l/m)}{\sin^2(\pi/m)}\left(1 - \frac{m}{2\pi}\sin\left(\frac{2\pi}{m}\right)\right), \quad \Delta = \cos\left(\frac{\pi l}{m}\right)$$



TP, PRL **106** (2011); Ilievski and TP, CMP **318** (2013);
TP and Ilievski, PRL **111** (2013)

Another type of quasilocal conserved operators

. . . exists even in the isotropic XXX spin 1/2 chain!

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Theorem [Ilievski, Medenjak, TP, arXiv:1506.05049]

Traceless operators $X_s(t)$, $s \in \frac{1}{2}\mathbb{Z}$, $t \in \mathbb{R}$, defined as

$$\begin{aligned} X_s(t) &= [\tau_s(t)]^{-n} \left\{ T_s(-\tfrac{1}{2} + \mathrm{i}t) T'_s(\tfrac{1}{2} + \mathrm{i}t) \right\}, \\ \tau_s(t) &= -t^2 - \left(s + \tfrac{1}{2}\right)^2, \end{aligned}$$

where $T_s(\lambda) = \mathrm{tr}_{\mathrm{a}} \mathbf{L}(\lambda, s)^{\otimes_{\mathrm{x}} n}$, $T'_s(\lambda) \equiv \partial_\lambda T_s(\lambda)$,
are quasilocal for all s, t and linearly independent from $\{Q_m; m \geq 1\}$ for $s > \frac{1}{2}$

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...and it resolves (!) the 2014–controversy with GGE
(Wouters et al. PRL 2014, Pozsgay et al. PRL 2014)
see:
(Ilievski, De Nardis, Wouters, Caux, Essler, TP, arXiv:1507.02993)

TP, PRL **112** (2014)

Extremely boundary driven open fermi Hubbard chain

$$H_n = \sum_{j=1}^{n-1} (\sigma_j^+ \sigma_{j+1}^- + \tau_j^+ \tau_{j+1}^- + \mathrm{H.c.}) + \frac{u}{4} \sum_{j=1}^{n} \sigma_j^z \tau_j^z$$

$$L_1 = \sqrt{\varepsilon}\sigma_1^+, \; L_2 = \sqrt{\varepsilon}\tau_1^+, \; L_3 = \sqrt{\varepsilon}\sigma_n^-, \; L_4 = \sqrt{\varepsilon}\tau_n^- .$$

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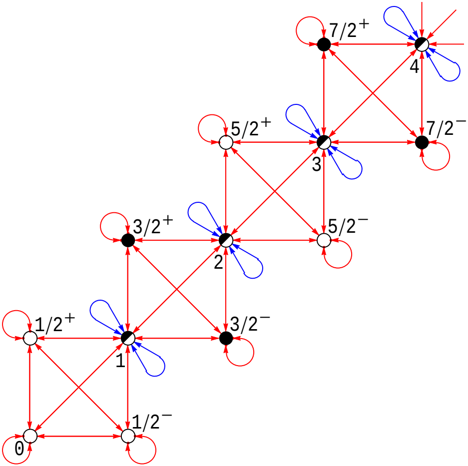
Key ansatz for NESS density matrix

$$\hat{\mathcal{L}}\rho_\infty = 0,$$

again a decomposition a-la Cholesky:

$$\rho_\infty = \Omega_n\Omega_n^\dagger.$$



$$\Omega_n = \sum_{\underline{e} \in \mathcal{W}_n(0,0)} a_{e_1} a_{e_2} \cdots a_{e_n} \prod_{j=1}^{n} \sigma_j^{b^{\mathrm{x}}(e_j)} \tau_j^{b^{\mathrm{y}}(e_j)}.$$

Popkov and TP, PRL **114**, 127201 (2015)

Amplitude Lindblad equation can again be reduced to LOD

$$[h_{x,x+1}, \mathbf{L}_{x,\mathrm{a}}\mathbf{L}_{x+1,\mathrm{a}}] = \tilde{\mathbf{L}}_{x,\mathrm{a}}\,\mathbf{L}_{x+1,\mathrm{a}} - \mathbf{L}_{x,\mathrm{a}}\,\tilde{\mathbf{L}}_{x+1,\mathrm{a}}$$

plus appropriate boundary conditions for the dissipators, solved with:

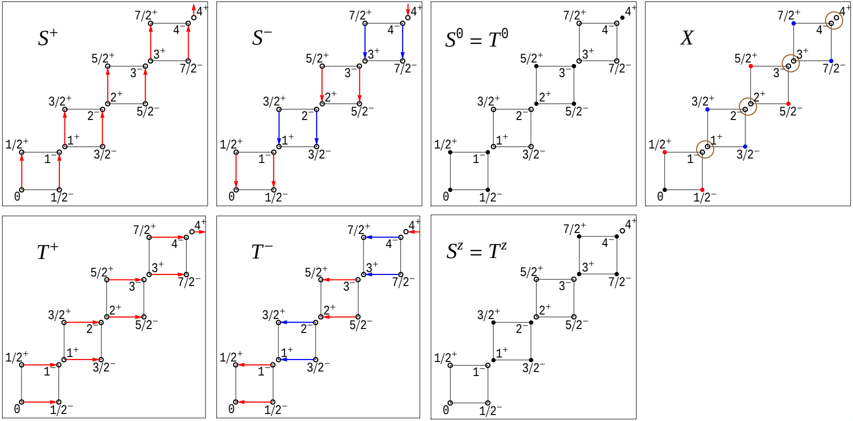
$$\Omega_n(\varepsilon) = \langle 0|_{\mathrm{a}}\mathbf{L}_{1,\mathrm{a}}\mathbf{L}_{2,\mathrm{a}}\cdots\mathbf{L}_{n,\mathrm{a}}|0\rangle_{\mathrm{a}},$$

again forming a commuting family

$$[\Omega_n(\varepsilon), \Omega_n(\varepsilon')] = 0.$$

Lax operator admits very appealing factorization

$$\mathbf{L}_{x,\mathrm{a}}(\varepsilon) = \sum_{\alpha,\beta\in\{+,-,0,\mathrm{z}\}} \mathbf{S}^{\alpha}_{\mathrm{a}}\mathbf{T}^{\beta}_{\mathrm{a}}\mathbf{X}_{\mathrm{a}}(u,\varepsilon)\sigma^{\alpha}_{x}\tau^{\beta}_{x}$$



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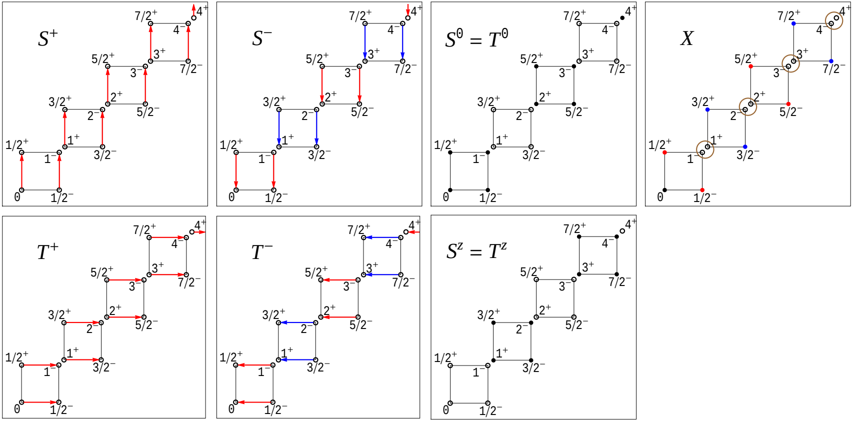

$\mathbf{S}^{\alpha}_{\mathrm{a}}$ and $\mathbf{T}^{\beta}_{\mathrm{a}}$ are particular commuting ($[\mathbf{S}^{\alpha}_{\mathrm{a}},\mathbf{T}^{\beta}_{\mathrm{a}}]=0$) representations of an extended CAR:

$$\{\mathbf{S}^{+},\mathbf{S}^{-}\} = 2(\mathbf{S}^{0}-\mathbf{S}^{\mathrm{z}}), \quad [\mathbf{S}^{\alpha},\mathbf{S}^{0}] = [\mathbf{S}^{\alpha},\mathbf{S}^{\mathrm{z}}] = 0.$$

Let the lattice of $n+1$ sites be $\Lambda_n \equiv [-n/2, n/2]$. Then:

$$Q_{\Lambda_n} = \mathrm{i}(Z_n - Z_n^\dagger)$$

where $Z_n = \mathrm{i}(\mathrm{d}/\mathrm{d}\varepsilon)\Omega_n|_{\varepsilon=0}$

$$Z_n = -\frac{1}{2}\sum_{x=-n/2}^{n/2-1}(\sigma_x^+\sigma_{x+1}^- + \tau_x^+\tau_{x+1}^-) + \frac{u}{2}\sum_{x,y\in\Lambda_n}^{x<y}(-1)^{x-y}\sigma_x^+ P^{(\sigma)}_{x+1,y-1}\sigma_y^-\tau_x^+ P^{(\tau)}_{x+1,y-1}\tau_y^-,$$

$P^{(\sigma)}_{x,y} := \sigma^{\mathrm{z}}_x\sigma^{\mathrm{z}}_{x+1}\cdots\sigma^{\mathrm{z}}_y$, $P^{(\tau)}_{x,y} := \tau^{\mathrm{z}}_x\tau^{\mathrm{z}}_{x+1}\cdots\tau^{\mathrm{z}}_y$, and $P^{(\sigma,\tau)}_{x,y} \equiv \mathbb{1}$ if $x > y$, satisfying the almost conservation condition

$$[H_{\Lambda_n}, Q_{\Lambda_n}] = \frac{1}{2}(\sigma^{\mathrm{z}}_{-n/2} + \tau^{\mathrm{z}}_{-n/2} - \sigma^{\mathrm{z}}_{n/2} - \tau^{\mathrm{z}}_{n/2}).$$

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Q_{Λ_n} is **quadratically extensive**, as $\langle Q_{\Lambda_n}^2\rangle \to qn^2$

Strict lower bounds on Green-Kubo diffusion constants in terms of quadratically extensive almost conserved quantities PRE **89**,012142(2014)

Theorem: *Take an arbitrary self-adjoint local current density operator j, satisfying $\langle j \rangle = 0$, where $\langle \bullet \rangle$ is an infinite temperature (tracial) state, and define a spatiotemporal correlation function of the infinite lattice dynamics as*

$$C(x,t) = \lim_{n\to\infty} \langle j(0,0) j(x,t) \rangle. \tag{1}$$

Assuming that $C(t) := \sum_{x=-\infty}^{\infty} C(x,t)$ exists for any t, that $D := \int_{-\infty}^{\infty} \mathrm{d}t\, C(t)$ and $D' := \int_{-\infty}^{\infty} \mathrm{d}t\, |t| C(t)$ exist as well, and that Q_{Λ_n} has a well defined component along j, $Q^j := \lim_{n\to\infty} \langle j Q_{\Lambda_n} \rangle$, the following inequality holds

$$D \geq \frac{|Q^j|^2}{8vq}. \tag{2}$$

where v is the Lieb-Robinson group velocity.

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For example, for the Hubbard chain, our bound evaluates to

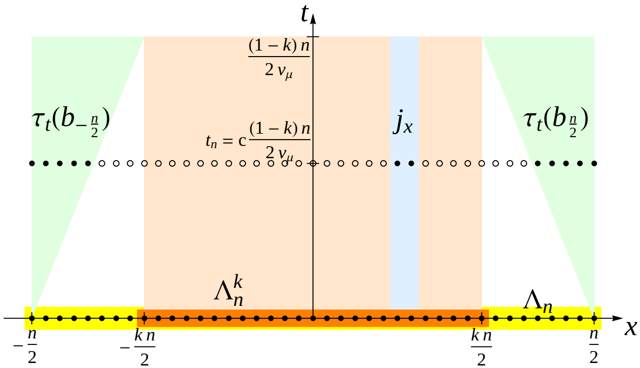
$$D^{\mathrm{c,s}} \geq \frac{2}{3u^2}.$$

for spin and charge currents
$j^{\mathrm{c,s}} = -2\mathrm{i} \left[\sigma^+ \otimes \sigma^- - \sigma^- \otimes \sigma^+ \pm (\tau^+ \otimes \tau^- - \tau^- \otimes \tau^+) \right]$ satisfying continuity equations $\mathrm{i}[H_{\Lambda_n}, \sigma^{\mathrm{z}}_x \pm \tau^{\mathrm{z}}_x] = j^{\mathrm{c,s}}_x - j^{\mathrm{c,s}}_{x-1}$. (Agrees with DMRG numerics!)

$$A_{n,t} := \frac{1}{t}\int_0^t \mathrm{d}t' \left(\tau_{t'}(J_{\Lambda_n^k}) - \frac{\alpha}{n} Q_{\Lambda_n} \right)$$

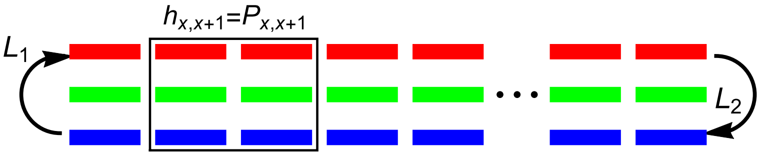
Since $A_{n,t}^2 \geq 0$, we have $\langle A_{n,t}^2 \rangle \geq 0$ for any $t = t_n, \alpha \in \mathbb{R}$, $n \in \mathbb{Z}^+$:

$$\int_0^{t_n} \mathrm{d}t' \int_0^{t_n} \mathrm{d}t'' \frac{1}{t^2} \langle \tau_{t'}(J_{\Lambda_n^k}) \tau_{t''}(J_{\Lambda_n^k}) \rangle - \int_0^{t_n} \mathrm{d}t' \frac{2\alpha}{nt} \langle \tau_{t'}(J_{\Lambda_n^k}) Q_{\Lambda_n} \rangle + \frac{\alpha^2}{n^2} \langle Q_{\Lambda_n}^2 \rangle \geq 0.$$



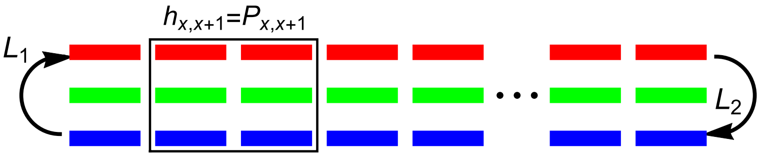
Lai-Sutherland $SU(3)$ model with degenerate dissipative driving

Ilievski and TP, NPB **882** (2014)



$$H = \sum_{x=1}^{n-1} h_{x,x+1}$$

`Ilievski and TP, NPB` **882** `(2014)`



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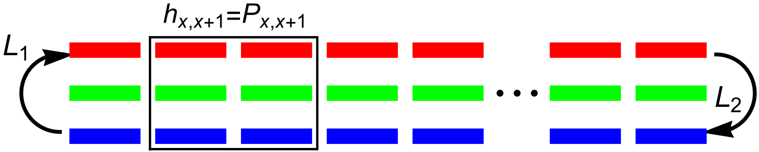
Ilievski and TP, NPB **882** (2014)



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Again, Cholesky factor of NESS has a Lax form, where the auxiliary space now needs 2 oscillator modes and a complex spin (basis labelled by inf. $3D$ lattice).

NESS manifold is infinitely degenerate (in TD limit). NESSes can be labeled by the *fixed number of green particles* ('doping'/chemical potential in TDL).

Closing with a list of open problems

- Why only "pure *particle-source* on one-end and pure *particle-sink* on the other-end" boundary conditions seem to be generally exactly solvable?
- More general boundary conditions need to be discussed. Perhaps generalising the notion of Sklyanin Reflection Algebra.
- The interpretation in terms of quantum symmetries are missing for some solutions, e.g. of open Hubbard chain.
  No apparent link to integralility structures (R& L-matrix) proposed by Shastry.
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