MATRIX ANSATZ FOR EXCLUSION PROCESSES

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Matrix ansatz and exclusion processes

Phase diagram of the TASEP

Correlation functions and Brownian excursions

Infinite line and shocks

Additivity and large deviations function of the density

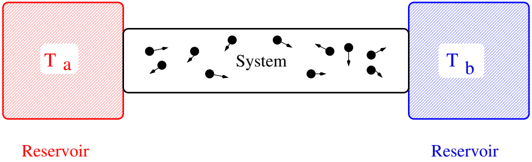
1993 $\rightarrow$ 2013
M. Evans, V. Hakim, V. Pasquier
S. Janowsky, J.L. Lebowitz, E.R. Speer
K. Mallick, D. Mukamel
C. Enaud, M. Retaux

Reviews

Richard A. Blythe and Martin R. Evans (2007)
Nonequilibrium steady states of matrix-product form: a solver's guide
Journal of Physics A: Mathematical and Theoretical, 40(46), R333

Alexandre Lazarescu (2015)
The Physicist's Companion to Current Fluctuations: One-Dimensional Bulk-Driven Lattice Gases
preprint arXiv:1507.04179

NON-EQUILIBRIUM STEADY STATES



Equilibrium $T_a = T_b = T$

$$P(\mathcal{C}) = Z^{-1} e^{-\frac{E(\mathcal{C})}{kT}}$$

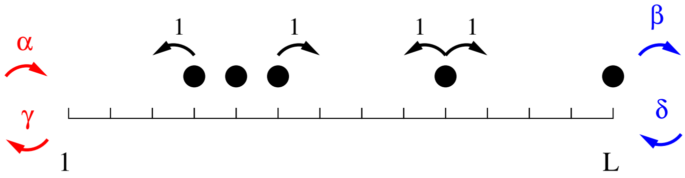
No phase transition in one dimension for short-range interactions

Non-equilibrium $T_a \neq T_b$

$$\boxed{P(\mathcal{C}) = ?}$$

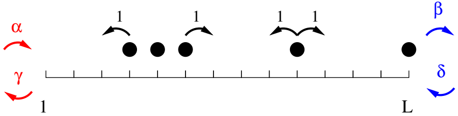
EXCLUSION PROCESSES

SSEP (Symmetric simple exclusion process)



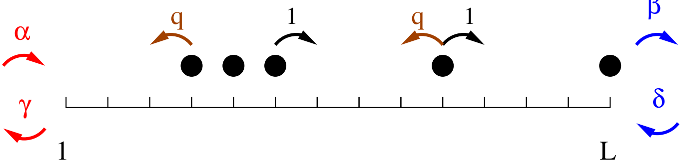
$$\rho_a = \frac{\alpha}{\alpha+\gamma}\ , \quad \tau_i = \begin{cases} 1 & \text{occupied} \\ 0 & \text{empty} \end{cases}\ , \quad \rho_b = \frac{\delta}{\beta+\delta}$$

SSEP

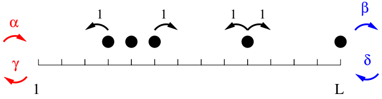


$$\rho_a = \frac{\alpha}{\alpha+\gamma} \;, \quad \tau_i = \begin{cases} 1 & \text{occupied} \\ 0 & \text{empty} \end{cases} \;, \quad \rho_b = \frac{\delta}{\beta+\delta}$$

ASEP (Asymmetric simple exclusion process)

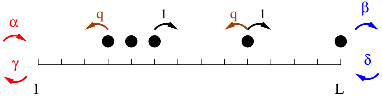


SSEP

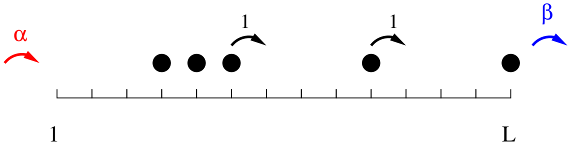


$$\rho_a = \frac{\alpha}{\alpha+\gamma}\,, \qquad \tau_i = \begin{cases} 1 & \text{occupied} \\ 0 & \text{empty} \end{cases}\,, \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

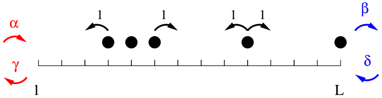
ASEP



TASEP (Totally ASEP)



STEADY STATE



$$\rho_a = \frac{\alpha}{\alpha+\gamma}\ , \quad \tau_i = \begin{cases} 1 & \text{occupied} \\ 0 & \text{empty} \end{cases}\ , \quad \rho_b = \frac{\delta}{\beta+\delta}$$

Equilibrium $\rho_a = \rho_b = \rho$

$$P(\tau_1, \ldots \tau_L) = \prod_{i=1}^{L} [\, \rho\, \tau_i + (1-\tau_i)(1-\rho)\,]$$

Non-equilibrium $\rho_a \neq \rho_b$

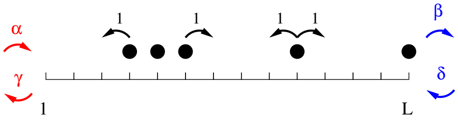
$P(\tau_1, \ldots \tau_L)$ given by the matrix ansatz

Alternative expressions: D. Domany Mukamel 1992, Schutz Domany 1993, Liggett 99

MATRIX ANSATZ

Fadeev 1980, ..., D. Evans Hakim Pasquier 1993

SSEP

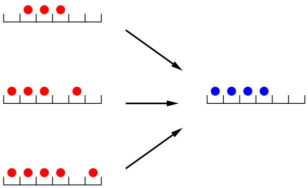


$$P(\tau_1, \ldots, \tau_L) = \frac{\langle W | X_1 \ldots X_L | V \rangle}{\langle W | (D+E)^L | V \rangle}$$

where $X_i = \begin{cases} D & \text{if site } i \text{ occupied} \\ E & \text{if site } i \text{ empty} \end{cases}$

$$\begin{aligned} \langle W | (\alpha E - \gamma D) &= \langle W | \\ DE - ED &= D + E \\ (\beta D - \delta E) | V \rangle &= | V \rangle \end{aligned}$$

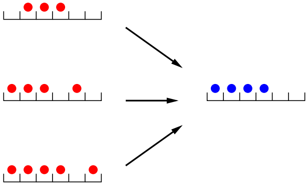
PROOF (SSEP)



$$\text{Gain} = \frac{\alpha\langle W|ED^3E^2|V\rangle + \langle W|D^3EDE|V\rangle + \beta\langle W|ED^4E|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

$$\text{Loss} = \frac{(\gamma+1+\delta)\langle W|D^4E^2|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

PROOF (SSEP)

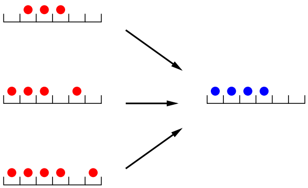


$$\text{Gain} = \frac{\alpha\langle W|ED^3E^2|V\rangle + \langle W|D^3EDE|V\rangle + \beta\langle W|ED^4E|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

$$\text{Loss} = \frac{(\gamma+1+\delta)\langle W|D^4E^2|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

$$\text{Gain} - \text{Loss} = \frac{\langle W|(\alpha E - \gamma D)D^3E^2 - D^3(DE - ED)E + D^4E(\beta D - \delta E)|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

PROOF (SSEP)



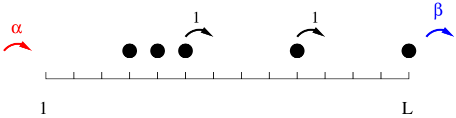
$$\text{Gain} = \frac{\alpha\langle W|ED^3E^2|V\rangle + \langle W|D^3EDE|V\rangle + \beta\langle W|ED^4E|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

$$\text{Loss} = \frac{(\gamma+1+\delta)\langle W|D^4E^2|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

$$\text{Gain} - \text{Loss} = \frac{\langle W|(\alpha E - \gamma D)D^3E^2 - D^3(DE - ED)E + D^4E(\beta D - \delta E)|V\rangle}{\langle W|(D+E)^6|V\rangle}$$

$$\boxed{\langle W|(\alpha E - \gamma D) = \langle W|} \quad \boxed{DE - ED = D + E} \quad \boxed{(\beta D - \delta E)|V\rangle = |V\rangle}$$

PHASE DIAGRAM FOR THE TASEP



$$\boxed{P(\tau_1, \ldots, \tau_L) = \frac{\langle W|X_1 \ldots X_L|V\rangle}{\langle W|(D+E)^L|V\rangle}}$$

where $X_i = D$ (site i occupied) and $X_i = E$ (empty)

$$\boxed{\begin{aligned} \langle W|\alpha E &= \langle W| \\ DE &= D + E \\ \beta D|V\rangle &= |V\rangle \end{aligned}}$$

Current through bond $i, i+1$

$$J = \frac{\langle W|(D+E)^{i-1}DE(D+E)^{L-i}|V\rangle}{\langle W|(D+E)^L|V\rangle} = \boxed{\frac{\langle W|(D+E)^{L-1}|V\rangle}{\langle W|(D+E)^L|V\rangle}}$$

$$\boxed{\begin{aligned} \langle W|\alpha E &= \langle W| \\ DE &= D + E \\ \beta D|V\rangle &= |V\rangle \end{aligned}}$$

If $F(E)$ is a polynomial, one has $\quad DF(E) = F(1)D + E\,\frac{F(E)-F(1)}{E-1}$

$$(D+E)^N = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!}\left(E^p + E^{p-1}D + \ldots + D^p\right)$$

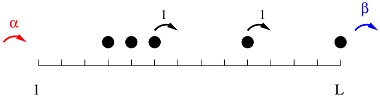
and

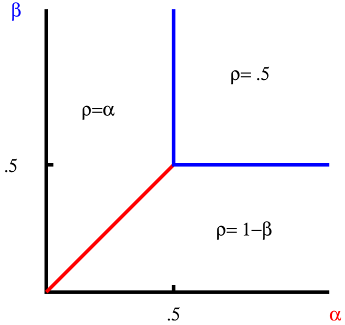
$$\frac{\langle W|E^m D^n|V\rangle}{\langle W|V\rangle} = \frac{1}{\alpha^m}\,\frac{1}{\beta^n}$$

$$\boxed{\frac{\langle W|(D+E)^N|V\rangle}{\langle W|V\rangle} = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!}\,\frac{\dfrac{1}{\alpha^{p+1}} - \dfrac{1}{\beta^{p+1}}}{\dfrac{1}{\alpha} - \dfrac{1}{\beta}}}$$

PHASE DIAGRAM FOR THE TASEP

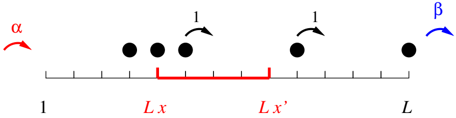
Krug 1991
D. Domany Mukamel 1992
D. Evans Hakim Pasquier 1993
Schutz Domany 1993





NON-GAUSSIAN DENSITY FLUCTUATIONS FOR THE ASEP

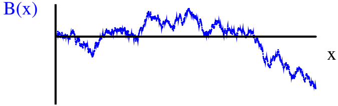
D. Enaud Lebowitz 2004

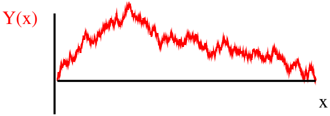


$N(x, x')$ number of particles between Lx and Lx'

$$\frac{N(x,x')}{L} - \frac{1}{2} = \frac{B(x') - B(x) + Y(x') - Y(x)}{2\sqrt{L}}$$

B is a Brownian path and Y is a Brownian excursion





TASEP: $\alpha = \beta = 1$

$$\langle W|\alpha E = \langle W| \qquad DE = D + E \qquad \beta D|V\rangle = |V\rangle$$

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ & & \ddots & \ddots & \end{pmatrix}$$

$$\text{with} \quad \langle W| = (1, 0, 0 \ldots) \qquad |V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

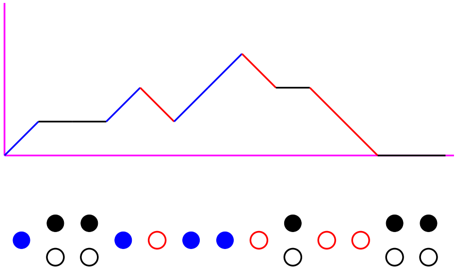
$$\frac{\langle W|(D+E)^N|V\rangle}{\langle W|V\rangle} = \sum_{\text{Paths}} \text{Weight(Path)}$$

$$\langle W| = (1, 0, 0 \ldots) \quad D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ & & \ddots & \ddots & \end{pmatrix} \quad |V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\boxed{\frac{\langle W|(D+E)^N|V\rangle}{\langle W|V\rangle} = \sum_{\text{Paths}} \text{Weight(Path)}}$$

$$\langle W| = (1,0,0\ldots) \quad D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ & & \ddots & \ddots & \end{pmatrix} \quad |V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\frac{\langle W|(D+E)^N|V\rangle}{\langle W|V\rangle} = \sum_{\text{Paths}} \text{Weight(Path)}$$



TASEP and Brownian excursions

$$\boxed{\left\langle \left(\tau_{Lx_1} - \frac{1}{2}\right) \dots \left(\tau_{Lx_k} - \frac{1}{2}\right) \right\rangle = \frac{1}{(4L)^{k/2}} \frac{d^k \ \langle y_1 \dots y_k \rangle}{dx_1 \dots dx_k}}$$

where $y(x)$ is a Brownian excursion and $y_i = y(x_i)$

small

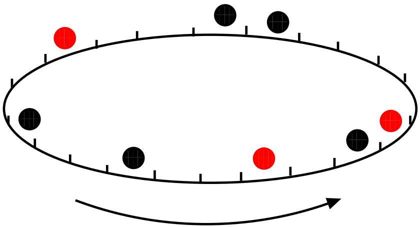
$$P(y_1 \dots y_k) = \frac{h_{x_1}(y_1)\, g_{x_2 - x_1}(y_1, y_2) \dots g_{x_k - x_{k-1}}(y_{k-1}, y_k)\, h_{1-x_k}(y_k)}{\sqrt{\pi}}$$

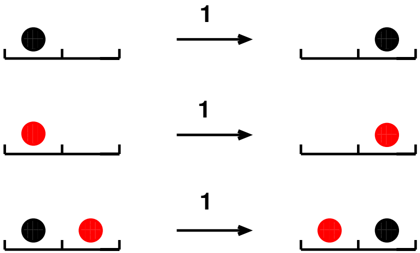
and

$$h_x(y) = \frac{2y}{x^{3/2}} e^{-y^2/x}$$

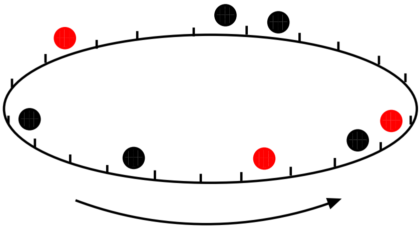
$$g_x(y, y') = \frac{1}{\sqrt{\pi x}} \left(e^{-(y-y')^2/x} - e^{-(y+y')^2/x} \right)$$

SECOND CLASS PARTICLE





SECOND CLASS PARTICLE

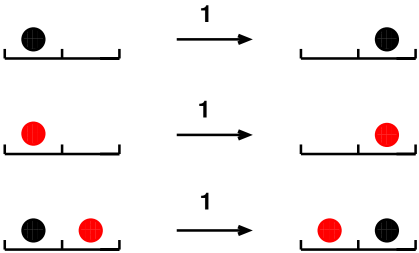


$\mathrm{tr}(X_1 X_2 \ldots X_N)$

First class $= D$

Second class $= A$

Empty $= E$

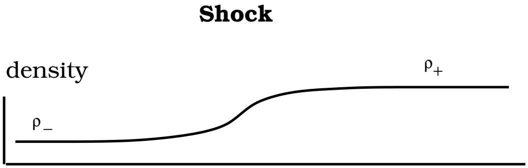


$$DE = D + E$$

$$DA = A$$

$$AE = A$$

Shocks



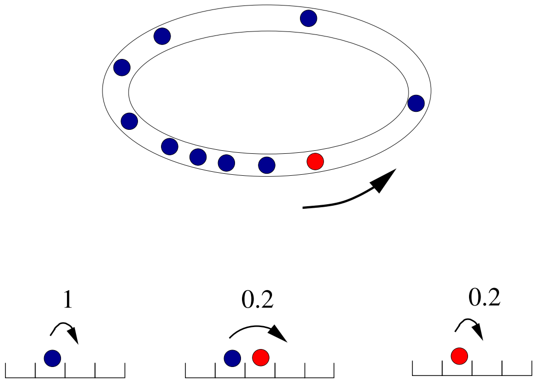
Weight= $\langle w|X_{-k}\cdots X_{-1}\ A\ X_1\cdots X_{k'}|v\rangle$

$$\langle w|(D+E)=\langle w| \quad ; \quad (D+E)|v\rangle=|v\rangle$$

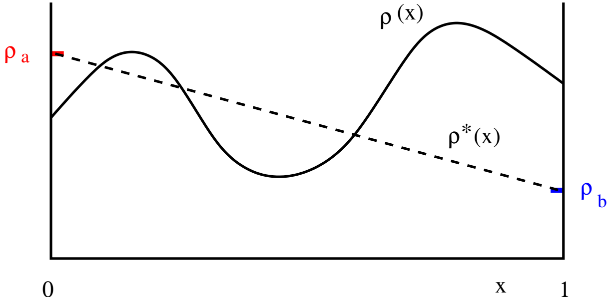
$$AE=(1-\rho_-)(1-\rho_+)A \quad ; \quad DA=\rho_-\rho_+A$$

$$DE=(1-\rho_-)(1-\rho_+)D+\rho_-\rho_+)E$$

PHASE TRANSITION



LARGE DEVIATION FUNCTIONAL $\mathcal{F}[\{\rho(x)\}]$

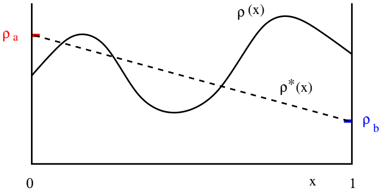


$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\mathcal{F}(\{\rho(x)\})]$$

For equilibrium systems, $\mathcal{F}(\{\rho(x)\})$ is the free energy

For the typical profile $\rho^*(x)$, one has $\mathcal{F}(\{\rho^*(x)\}) = 0$

LARGE DEVIATION FUNCTIONAL $\mathcal{F}[\{\rho(x)\}]$



$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\mathcal{F}(\{\rho(x)\})]$$

For equilibrium systems, $\mathcal{F}[\{\rho(x)\}]$ is the free energy

Procedure:

1. Cut the system into k boxes of length L/k
2. $\text{Pro}(\rho_1, \rho_2, ...\rho_k)$ is the sum of the weights of all microscopic configurations with densities ρ_1 in the first box, ... ρ_k in the kth box
3. For large L, $\text{Pro}(\rho_1, \rho_2, ...\rho_k) \sim \exp[-L\mathcal{F}(\rho_1, \rho_2, ...\rho_k)]$
4. Take $k \to \infty$ with $(k \ll L)$

LARGE DEVIATION FUNCTIONAL FOR THE SSEP

$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\mathcal{F}(\{\rho(x)\})]$$

Equilibrium $\rho_a = \rho_b = F$

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left[(1-\rho(x)) \log \frac{1-\rho(x)}{1-F} + \rho(x) \log \frac{\rho(x)}{F} \right]$$

Non-equilibrium $(\rho_a \neq \rho_b)$

D. Lebowitz Speer 2001-2002
Bertini De Sole Gabrielli Jona-Lasinio Landim 2002

$$\mathcal{F}(\{\rho(x)\}) = \sup_{F(x)} \int_0^1 dx \left[(1-\rho(x)) \log \frac{1-\rho(x)}{1-F(x)} + \rho(x) \log \frac{\rho(x)}{F(x)} + \frac{\log F'(x)}{\rho_b - \rho_a} \right]$$

with $F(x)$ monotone, $F(0) = \rho_a$ and $F(1) = \rho_b$

Non-equilibrium $(\rho_a \neq \rho_b)$

$$\mathcal{F}(\{\rho(x)\}) = \sup_{F(x)} \int_0^1 dx \left[(1-\rho(x)) \log \frac{1-\rho(x)}{1-F(x)} + \rho(x) \log \frac{\rho(x)}{F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a}\right]$$

with $F(x)$ monotone, $F(0) = \rho_a$ and $F(1) = \rho_b$

Consequences:

$\mathcal{F}$ is non-local: for example for small $\rho_a - \rho_b$

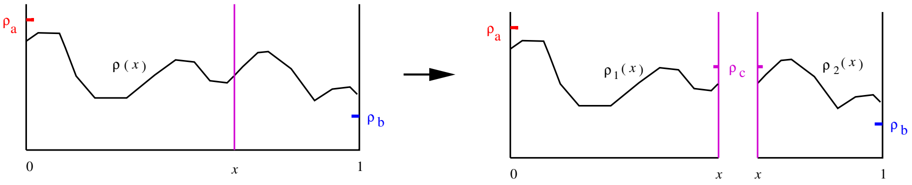
$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx\ (1-\rho(x)) \log \frac{1-\rho(x)}{1-\rho^*(x)} + \rho(x) \log \frac{\rho(x)}{\rho^*(x)}$$
$$+ \frac{(\rho_a - \rho_b)^2}{(\rho_a - \rho_a^2)^2} \int_0^1 dx \int_x^1 dy\, x(1-y)(\rho(x) - \rho^*(x))(\rho(y) - \rho^*(y))$$
$$+ O(\rho_a - \rho_b)^3$$

where $\rho^*(x) = \langle \rho(x) \rangle = (1-x)\rho_a + x\rho_b$

Long-range correlations   Spohn 82

$$\langle \rho(x)\rho(y) \rangle - \langle \rho(x) \rangle \langle \rho(y) \rangle \simeq \frac{1}{L} G(x,y) = -\frac{(\rho_a - \rho_b)^2}{L} x(1-y)$$

ADDITIVITY FOR THE SSEP



$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\mathcal{F}(\{\rho(x)\})]$$

Try to find ρ_c such that

$$\boxed{\mathcal{F}(\{\rho(x)\}|\rho_a, \rho_b) = x\ \mathcal{F}(\{\rho_1(x)\}|\rho_a, \rho_c) + (1-x)\ \mathcal{F}(\{\rho_2(x)\}|\rho_c, \rho_b)}$$

Idea

$$P(\tau_1, \ldots, \tau_L) = \frac{\langle W|X_1 \ldots X_L|V\rangle}{\langle W|(D+E)^L|V\rangle}$$

Try to insert a complete basis

$$\langle W|X_1 \ldots X_L|V\rangle = \int dU \; \langle W|X_1 \ldots X_{L'} \;\; |U\rangle K(U) \langle U| \;\; X_{L'+1} \ldots X_L|V\rangle$$

In practise

Define the eigenvectors $\langle \rho, a|$ and $|\rho, b\rangle$

$$\langle \rho, a| \; [\rho E - (1-\rho)D] = a\langle \rho, a|$$
$$[(1-\rho)D - \rho E] \; |\rho, b\rangle = b|\rho, b\rangle$$

Then $\langle W| = \langle \rho_a, (\alpha+\gamma)^{-1}|$ and $|V\rangle = |\rho_b, (\beta+\delta)^{-1}\rangle$

Then one can prove that:

$$\frac{\langle \rho_a, a|Y_1 Y_2|\rho_b, b\rangle}{\langle \rho_a, a|\rho_b, b\rangle} = \oint_{\rho_b < |\rho_c| < \rho_a} \frac{d\rho_c}{2i\pi} \frac{(\rho_a - \rho_b)^{a+b}}{(\rho_a - \rho_c)^{a+b}(\rho_c - \rho_b)} \frac{\langle \rho_a, a|Y_1|\rho_c, b\rangle}{\langle \rho_a, a|\rho_c, b\rangle} \frac{\langle \rho_c, 1-b|Y_2|\rho_b, b\rangle}{\langle \rho_c, 1-b|\rho_b, b\rangle}$$

LARGE DEVIATIONS OF THE DENSITY: EXTENSIONS

Large deviations of the density profile
D. Lebowitz Speer 2002-2003 ASEP
Enaud D. 2004 WASEP

Macroscopic fluctuation theory
Bertini De Sole Gabrielli Jona-Lasinio Landim 2001 $\rightarrow$
$\cdots$ KMP model
$\cdots$ tagged particle
$\cdots$ current fluctuations
$\cdots$ $\ldots$

CONCLUSION

Matrix ansatz for the steady state

Phase diagram

Correlation functions

Several species

Large deviation function

Finite size effects

Matrix ansatz for the current fluctuations

Talk by Kirone Mallick

Review by A. Lazarescu