Matrix Product State and Matrix Product Operator in Symmetry Protected Topological Phases

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Symmetric vs. Symmetry Breaking phases

- Transverse field Ising model

$$H = -\sum_{<ij>} Z_i Z_j + B \sum_i X_i$$

- Symmetry

$$\prod_i X_i \qquad \uparrow \Longleftrightarrow \downarrow$$

symmetry breaking

$$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$$

$$\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow$$

symmetric

$$\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$$

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle + |\downarrow\rangle$$

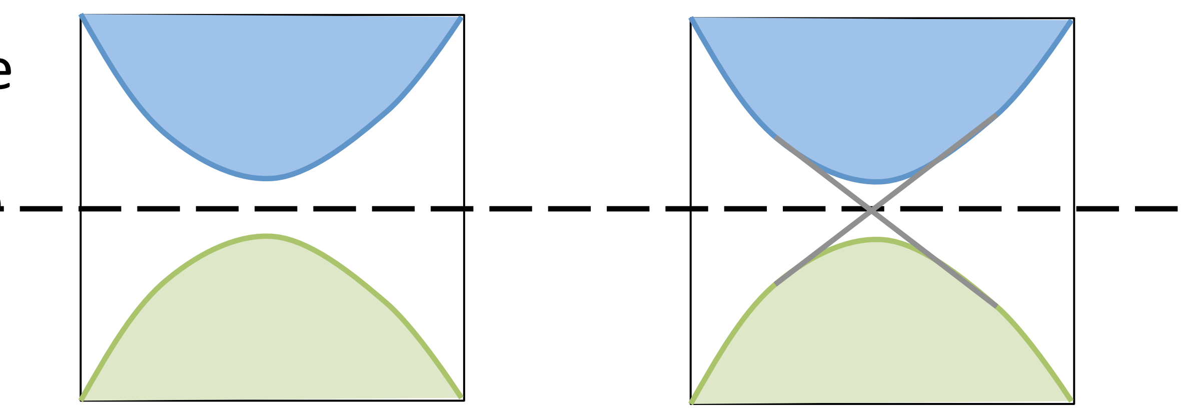
Different phases with the same symmetry

Kane, Mele (2005)

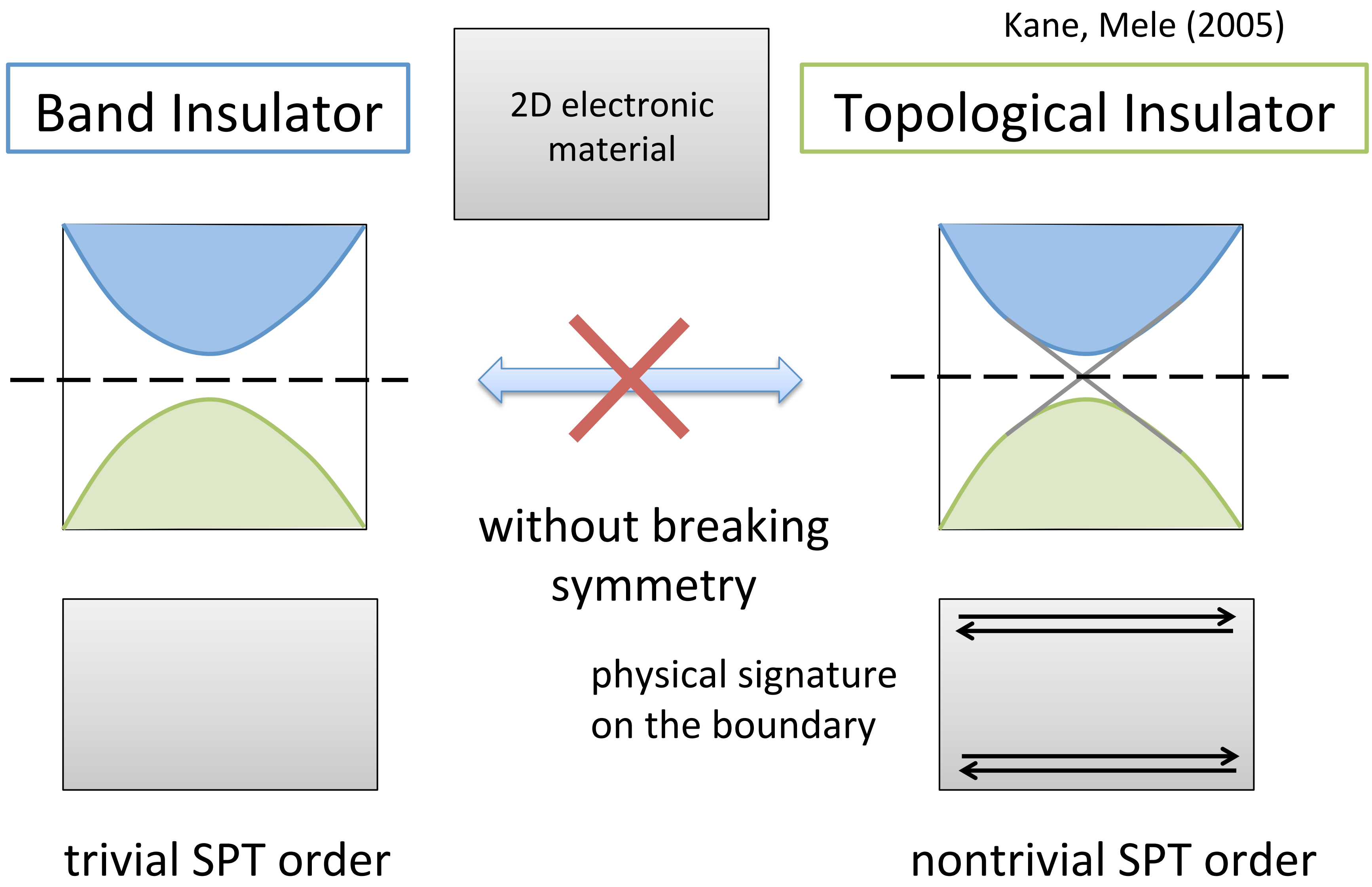
2D electronic material	Band Insulator	Topological Insulator
Symmetry	Charge conservation, time reversal	
Similarity	Insulating in the bulk	
Difference	Insulating on the boundary	Conducting on the boundary

Band structure

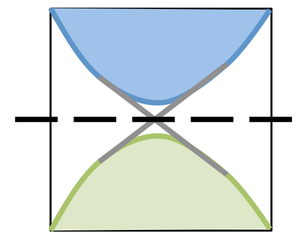
Fermi surface



Symmetry Protected Topological Order

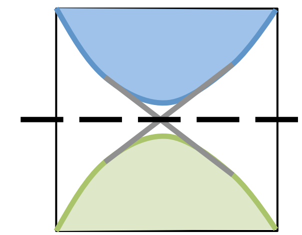


Symmetry Protected Topological Order

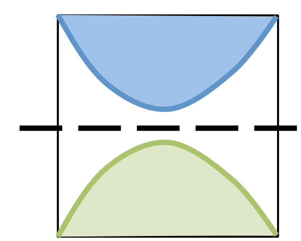


- Gapped quantum phases at zero T
- focus on ground state
- system has some symmetry
- ground state does not break symmetry
- Unusual property on the boundary (conducting channel, gapless modes which carry charge or energy)

Symmetry Protected Topological Order



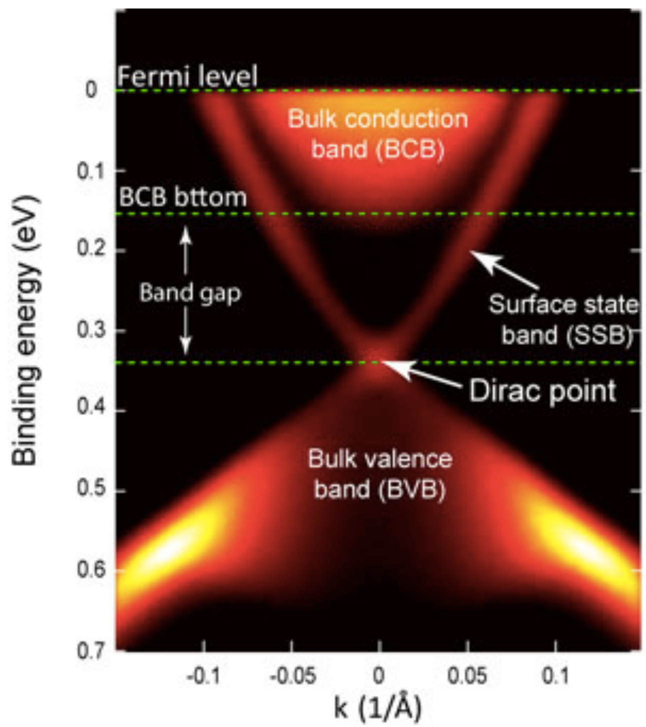
- Gapped quantum phases at zero T
- focus on ground state
- system has some symmetry
- If we explicitly break symmetry by adding symmetry breaking perturbations (e.g. adding magnetic impurities to break time reversal)
- Different SPT order can be smoothly connected

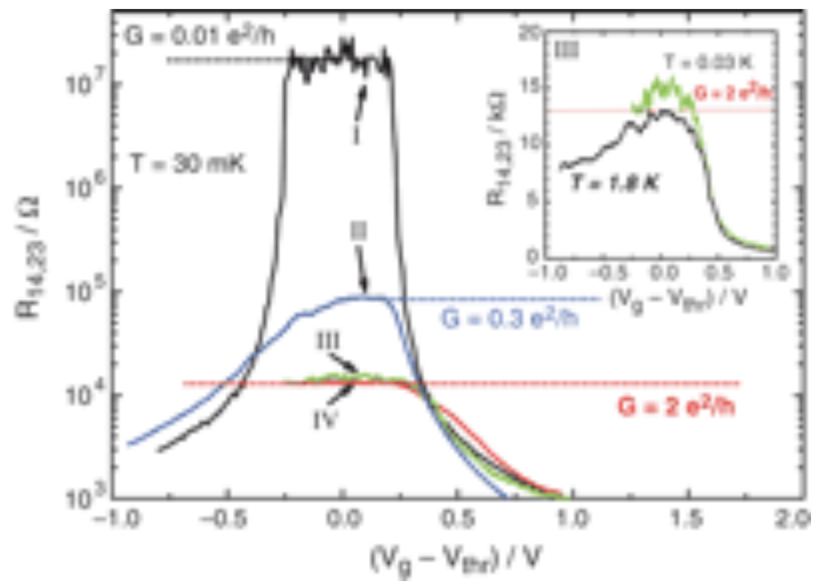


Topological Insulators

- Realization

Bernevig, Zhang (2006); Bernevig, Hughes, Zhang (2006); Konig et al (2007); Fu, Kane, Mele (2007); Moore, Balents (2007); Roy (2009); Hsieh, et al (2008); Chen, et al (2009);





- Classification

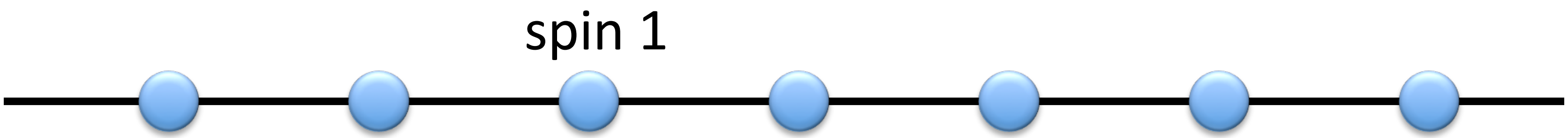
Kitaev (2009); Schnyder et al (2009)

Symmetry protected topological order in spin / boson systems?

- Gapped quantum phases at zero T
- system has some symmetry
- ground state does not break symmetry
- Unusual property on the boundary
- If symmetry is explicitly broken, can be smoothly connected to a trivial phase

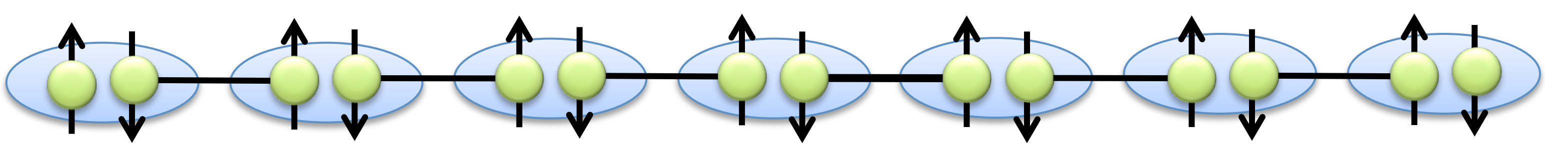
Example: Haldane phase and AKLT

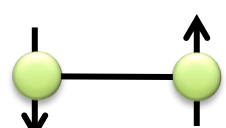
Haldane (1983); Affleck, Kennedy, Lieb, Tasaki, (1987)



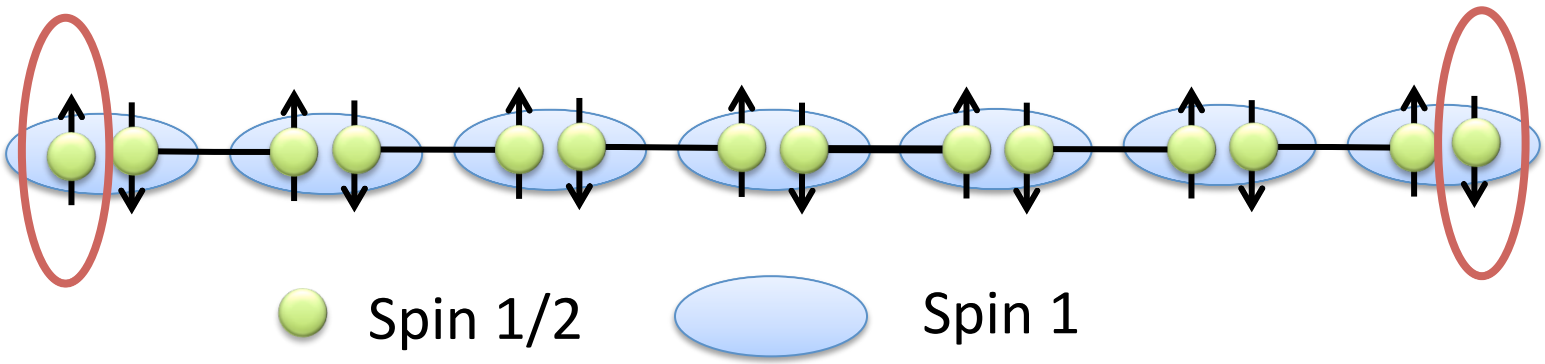
$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2$$



 Spin 1/2  Singlet $|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$

AKLT model



- Spin rotation symmetry
- Bulk gapped, does not break symmetry
- effective spin ½ degenerate edge state

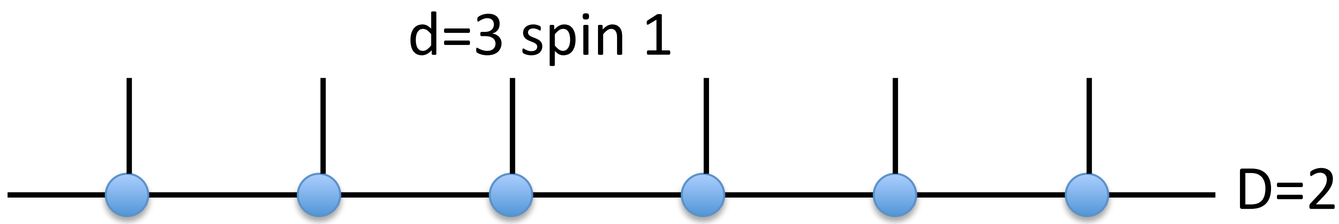
Affleck, Kennedy, Lieb, Tasaki, (1987)

Symmetry Protected Topological Order

Properties	Topological Insulator	AKLT chain
Has certain symmetry	Time reversal and charge conservation	Spin rotation
Ground state does not break symmetry	✓	✓
Gapped bulk	✓	✓
Gapless / degenerate edge	Gapless	Degenerate (spin ½)
Can be gapped by breaking symmetry	Magnetic field / superconductivity	Magnetic field

What other SPT phases exist in 1D spin chains?

Matrix Product State rep. of AKLT



$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_n} Tr(A^{i_1} A^{i_2} \ldots . A^{i_n}) |i_1 i_2 \ldots i_n\rangle$$

$$A_x = \sigma_x, A_y = \sigma_y, A_z = \sigma_z$$

$$|x\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle), \; |y\rangle = \frac{-i}{\sqrt{2}}(|1\rangle + |-1\rangle), \; |z\rangle = -|0\rangle$$

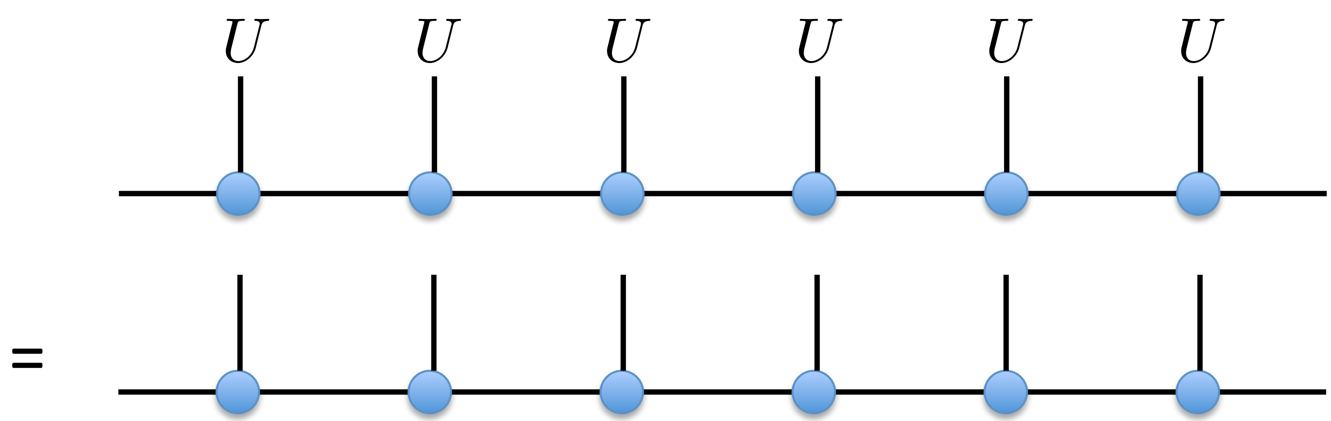
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

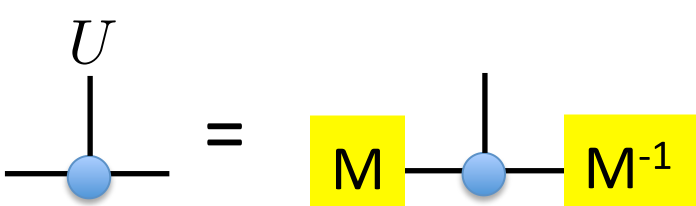
Affleck, Kennedy, Lieb, Tasaki, (1987); D. Perez-Garcia, F. Verstraete, M.M. Wolf, J.I. Cirac (2007)

MPS and symmetry

spin rotation symmetry $\prod U^k|\psi\rangle = |\psi\rangle$

$$U = e^{i\theta\vec{n}\cdot\vec{S}}$$

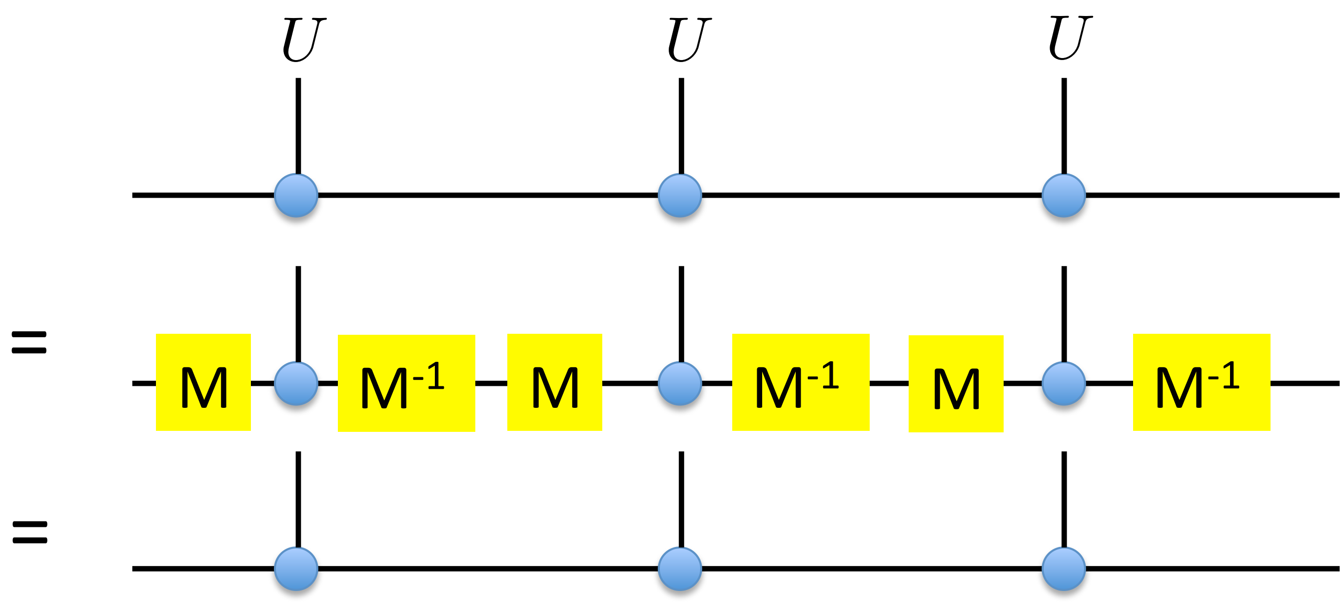


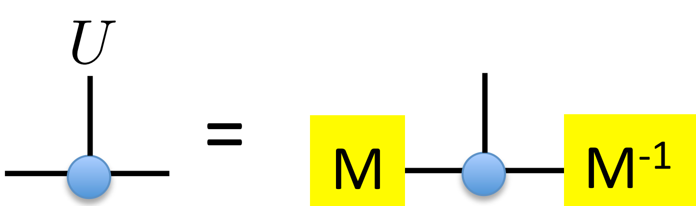


$$\sum_j U_{ij} A^j = M A^i M^{-1}$$

MPS and symmetry

symmetry $\prod U^k |\psi\rangle = |\psi\rangle$ $\quad U = e^{i\theta \vec{n} \cdot \vec{S}}$





$$\sum_j U_{ij} A^j = M A^i M^{-1}$$

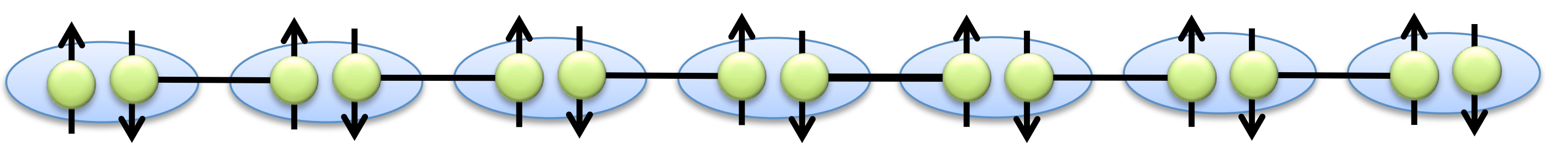
MPS and symmetry

AKLT

$$U = e^{i\theta\vec{n}\cdot\vec{S}} \qquad M = e^{i\theta\vec{n}\cdot\vec{\sigma}}$$

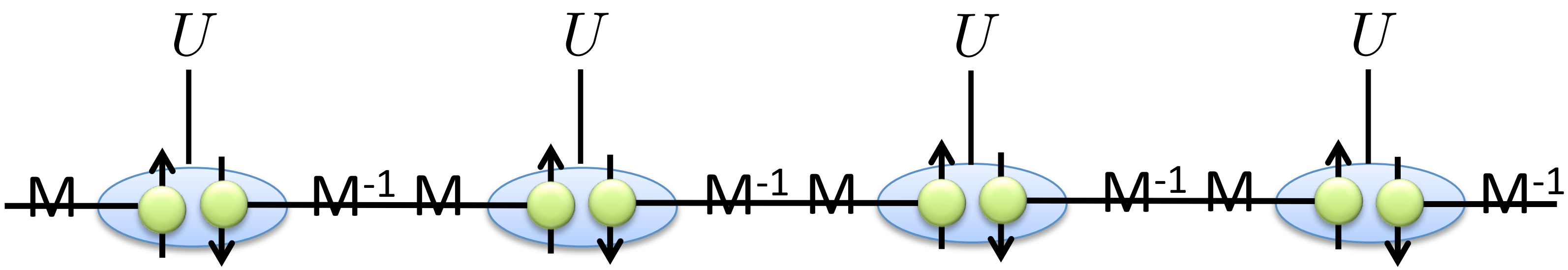
$\vec{S}$ spin 1 $\vec{\sigma}$ spin 1/2

Physical degree of freedom Effective edge degree of freedom



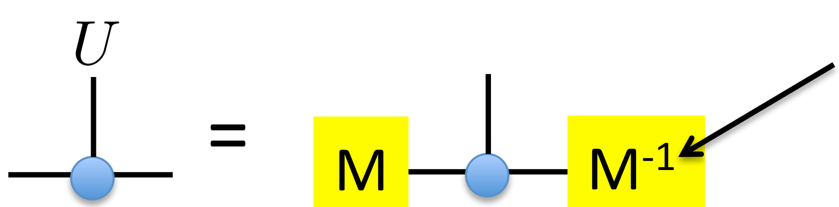
MPS and symmetry

AKLT



U rotation on spin 1

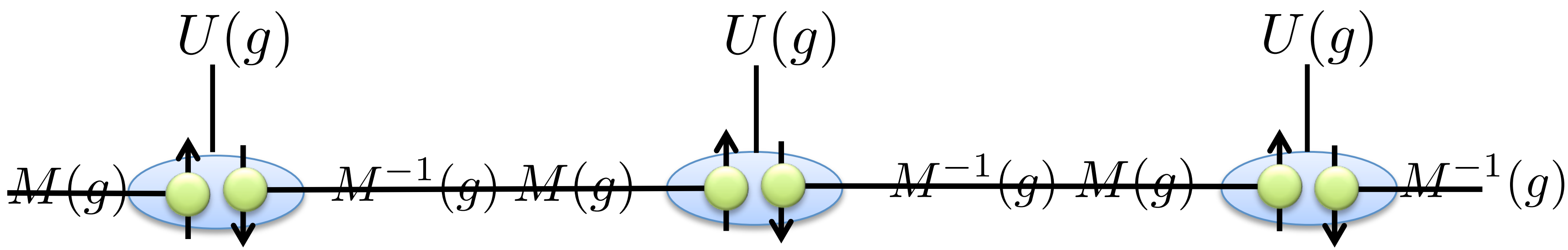
M rotation on spin 1/2



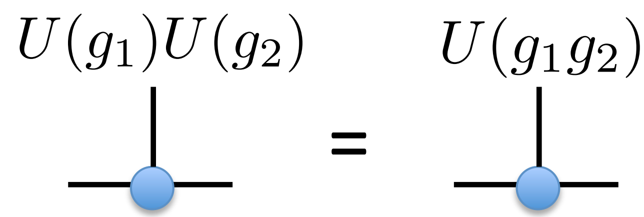
Symmetry Protected Topological Order

MPS and symmetry

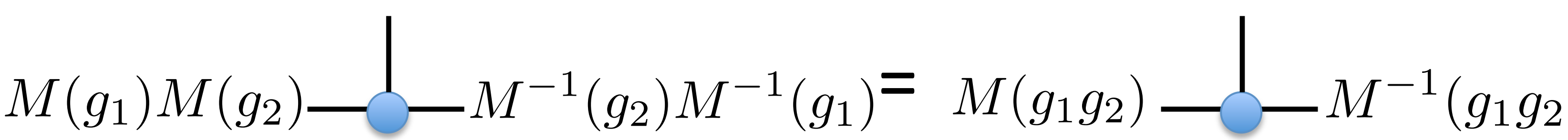
More generally symmetry group G $g \in G$



$$U(g_1)U(g_2) = U(g_1 g_2)$$



$$M(g_1)M(g_2) = \alpha M(g_1 g_2)$$



Fannes, Nachtergaele, R. F. Werner (1992); D. Perez-Garcia, F. Verstraete, M.M. Wolf, J.I. Cirac (2007); Pollmann, Berg ,Turner, Oshikawa, (2010)

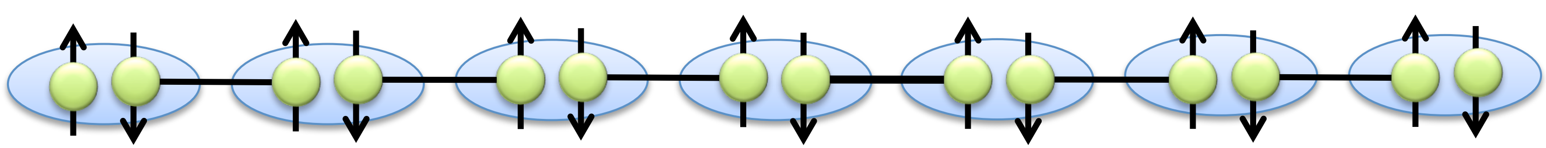
MPS and symmetry

- AKLT, spin rotation symmetry

$$U_{\vec{n}}(\pi)U_{\vec{n}}(\pi) = e^{i\pi\vec{n}\cdot\vec{S}}e^{i\pi\vec{n}\cdot\vec{S}} = I$$

$$M_{\vec{n}}(\pi)M_{\vec{n}}(\pi) = e^{i\pi\vec{n}\cdot\vec{\sigma}}e^{i\pi\vec{n}\cdot\vec{\sigma}} = -I$$

- Projective representation
  - different classes cannot be smoothly connected
  - must be > 1D, degeneracy

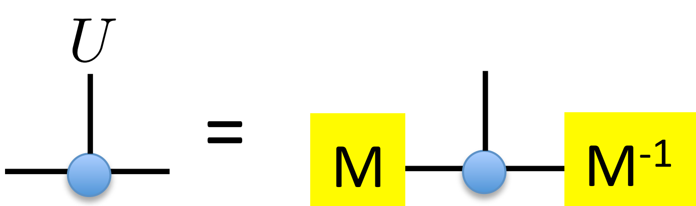


Pollmann, Berg ,Turner, Oshikawa, (2010)

MPS and general 1D SPT

All gapped ground states in 1D can be represented as MPS!

All gapped MPS transform under symmetry as



$$M(g_1)M(g_2) = \alpha M(g_1 g_2)$$

Symmetry Protected Topological Phase with symmetry G ⟺ Projective representation of group G

Verstraete, Cirac (2006); Hastings (2007); Fannes, Nachtergaele, R. F. Werner (1992); D. Perez-Garcia, F. Verstraete, M.M. Wolf, J.I. Cirac (2007); Turner, Pollmann, Berg (2010); **XC**, Gu, Wen (2011); Schuch, Perez-Garcia, Cirac (2011)

MPS and general 1D SPT

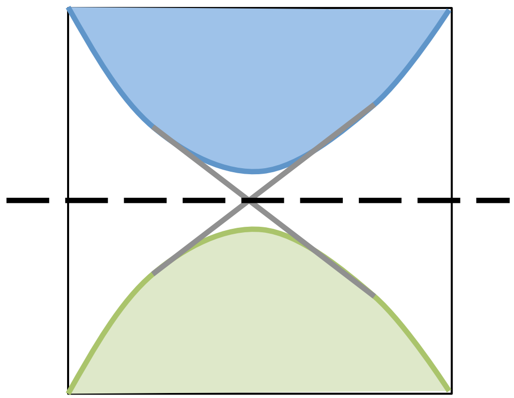
- Complete classification of phases in interacting systems

Symmetry	Z_2 x Z_2	Spin rotation	Time Reversal	Z_2
d=1	1	1	1	0

- Impossible to achieve by brute force
- Provide important numerical tool
- Starting point for generalization to higher dimensions

2D SPT in spin / boson systems

- Bosonic / spin version of topological insulator?
- Gapped quantum phases at zero T
- system has some symmetry
- ground state does not break symmetry
- Gapless boundary protected by symmetry



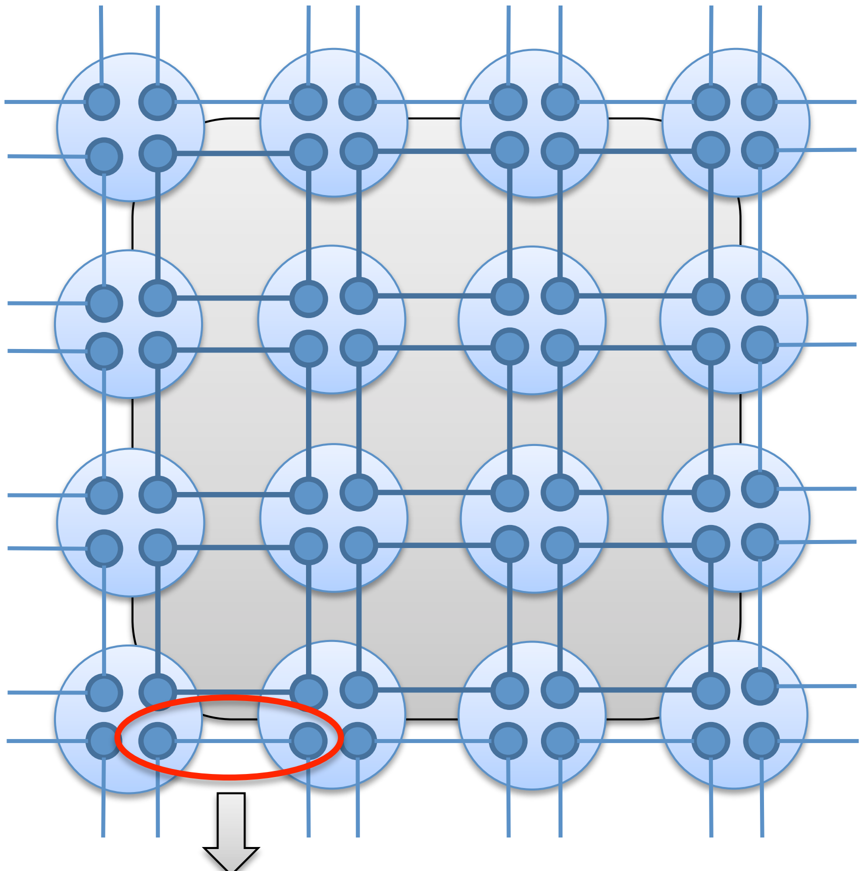
2D spin SPT order with Z_2 sym

- Exactly solvable model
- Bulk is gapped
- does not break symmetry
- Gapless boundary
  - need to show that the boundary is gapless under any symmetric interacting perturbation
  - matrix product operator

XC, Liu, Wen (2012)

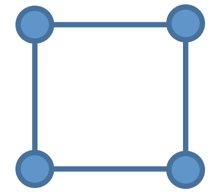
2D SPT order with Z_2 symmetry

XC, Liu, Wen (2012)

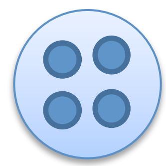


$$00, 11 \rightarrow \tilde{0}, \tilde{1}$$

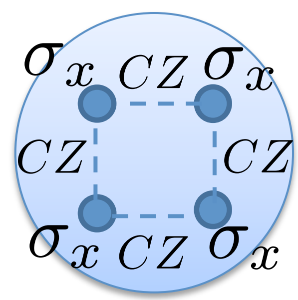
2 level system |0>, |1>



$$\left|\begin{matrix}0 & 0\\0 & 0\end{matrix}\right\rangle + \left|\begin{matrix}1 & 1\\1 & 1\end{matrix}\right\rangle$$



Global Onsite Z_2 symmetry

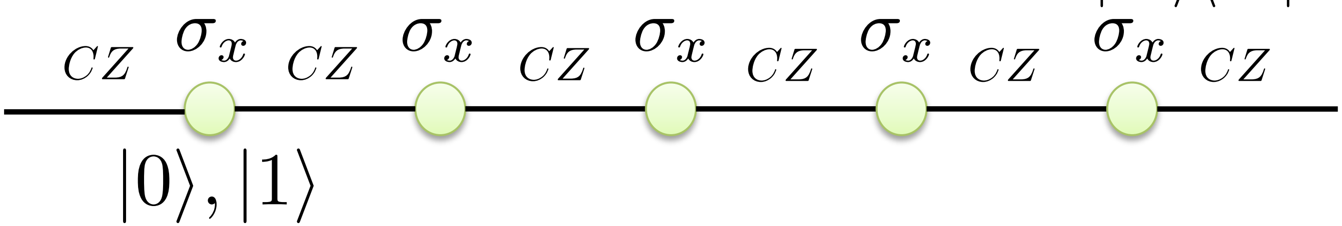


$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$

2D SPT order with Z_2 symmetry

- Effective boundary state
- Effective Z_2 symmetry

$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$



- Must be gapless unless symmetry is broken

$$H = -\sum_i \sigma_z^i \sigma_z^{i+1}$$ symmetry breaking

$$H = \sum_i \sigma_x^i + \sigma_z^{i-1}\sigma_x^i\sigma_z^{i+1}$$ gapless

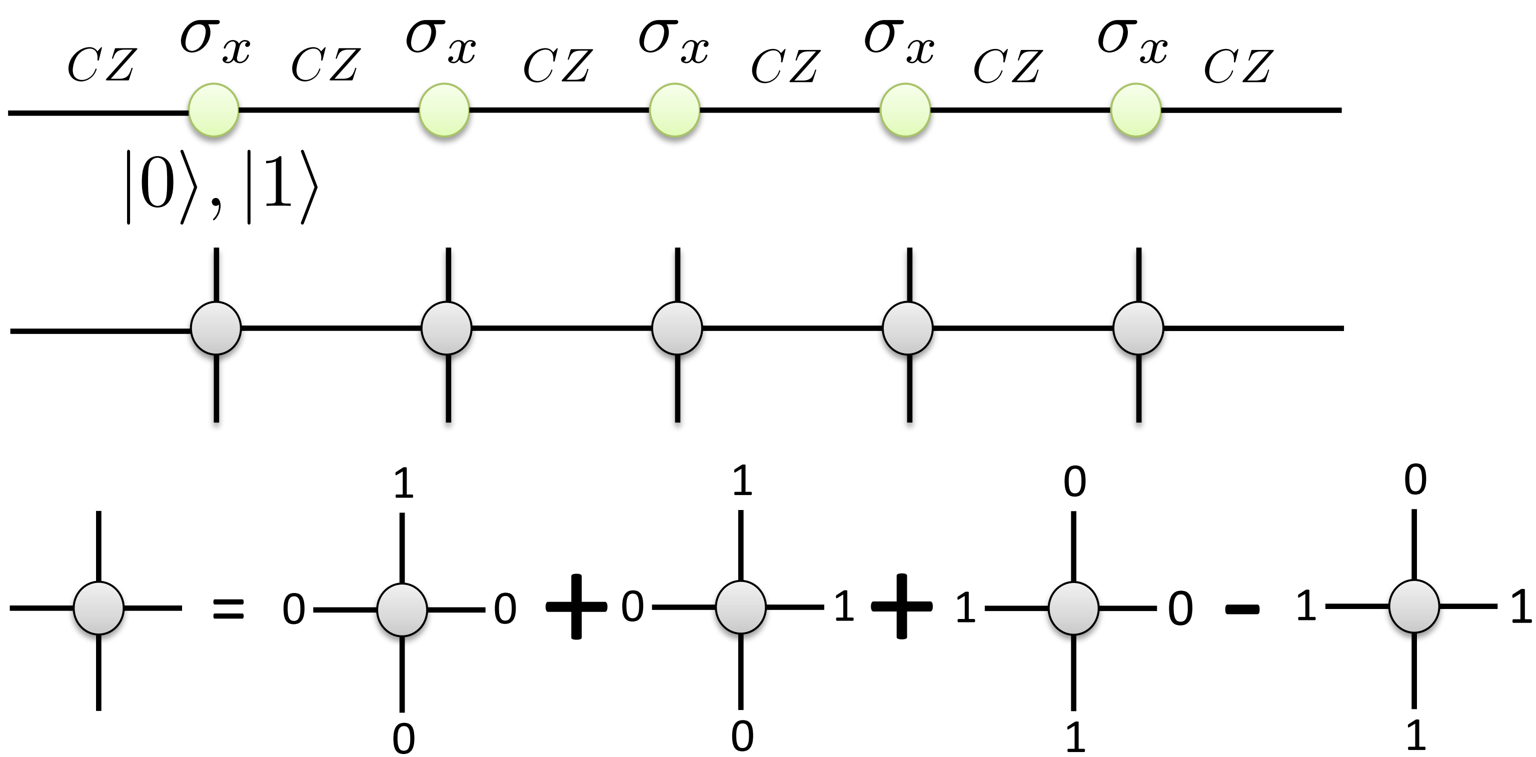
- Proven using matrix product state and operator

XC, Liu, Wen (2012)

2D SPT order with Z_2 symmetry

- Effective boundary state
- Effective Z_2 symmetry

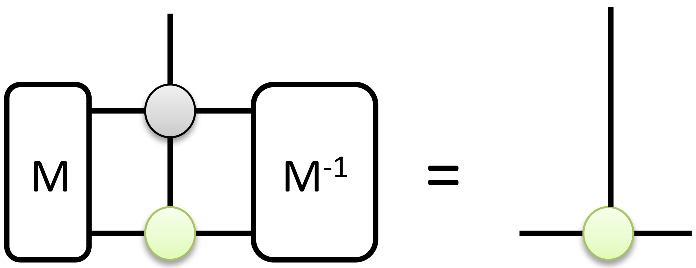
$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$



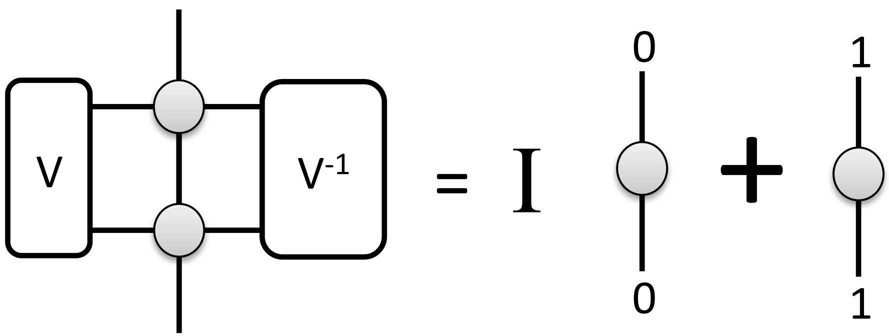
Murg, Cirac, Pirvu, Verstraete (2010)

2D SPT order with Z_2 symmetry

- If the boundary state can be both symmetric and gapped

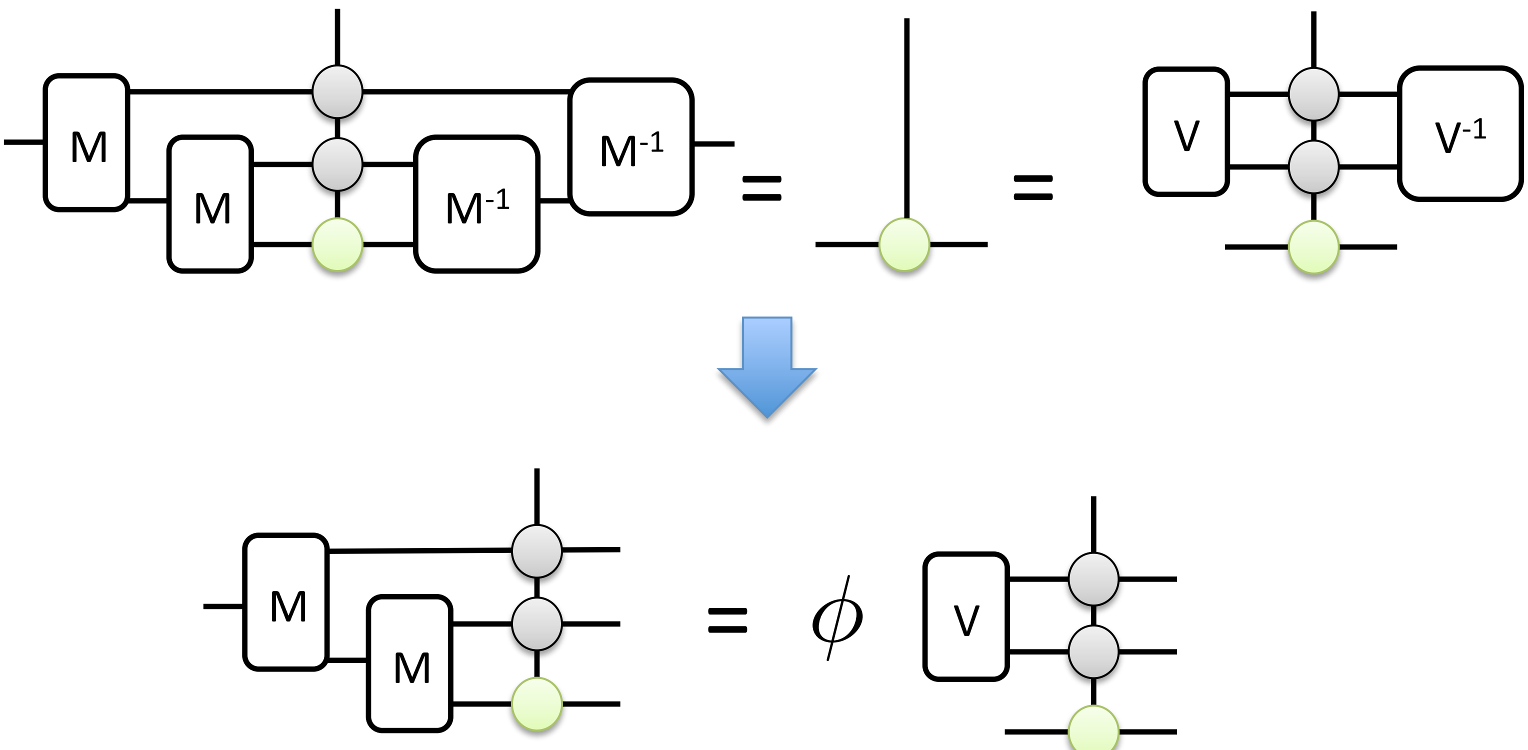


- Moreover

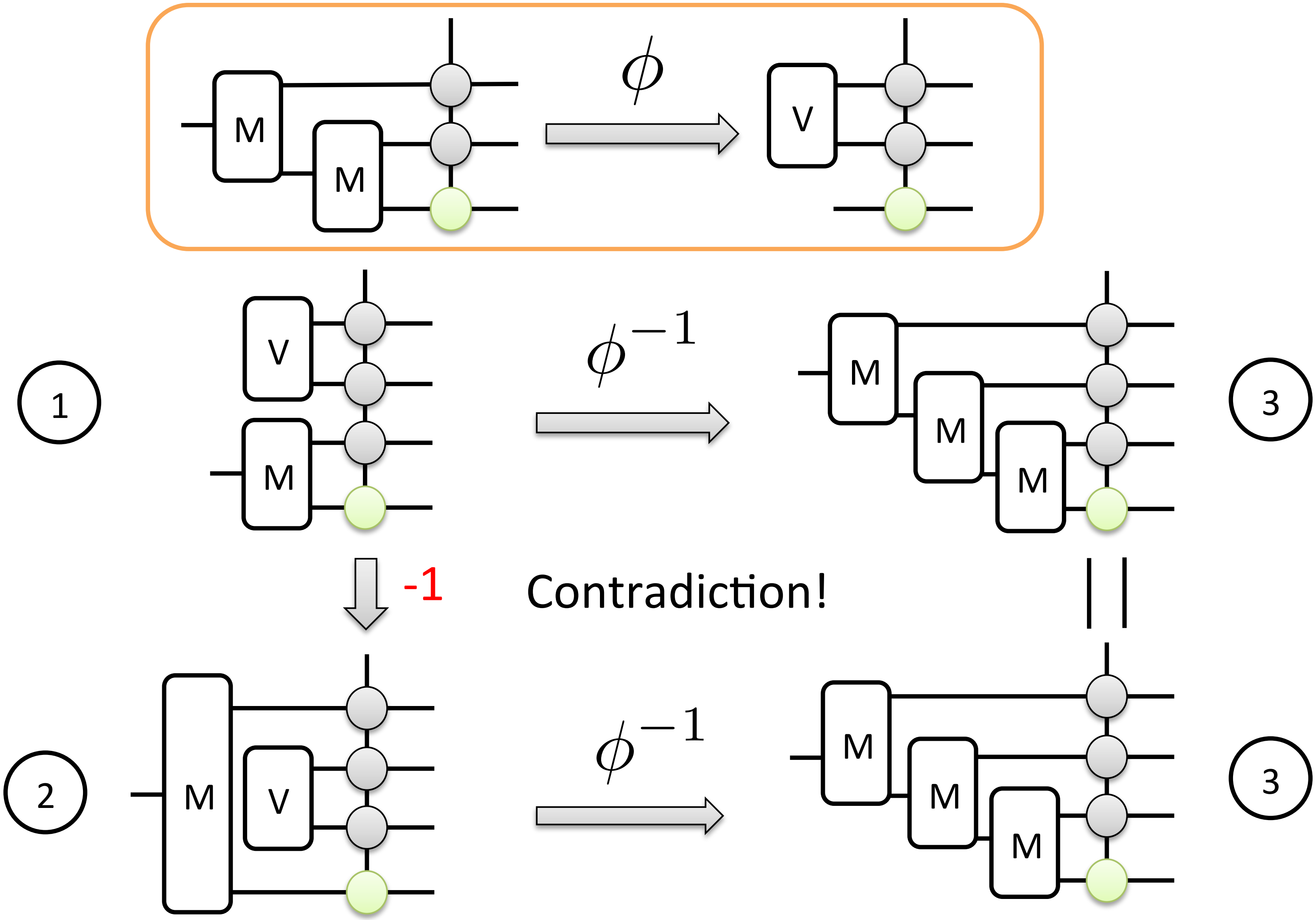


Not consistent

2D SPT order with Z_2 symmetry



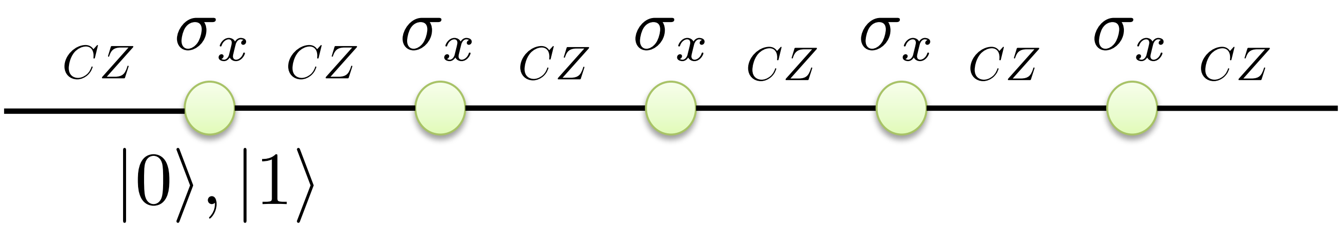
2D SPT order with Z_2 symmetry



2D SPT order with Z_2 symmetry

- Effective boundary state
- Effective Z_2 symmetry

$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$



- It is not possible to have a gapped boundary state without breaking symmetry

General 2D SPT with spins / bosons

- Systematic construction of 2D SPT with any G
- Proves the gapless-ness of symmetric boundary state
- Proves nontrivial SPT order in interacting systems

Symmetry	Z_2	Charge Conservation (C)	Time Reversal (T)	C & T
d=2	1	∞	0	1

XC, Gu, Liu, Wen (2012); **XC**, Gu, Liu, Wen (2013)

Summary

- In interacting spin / boson systems
- Matrix Product State and classification of 1D symmetry protected topological order
- Matrix Product Operator and systematic construction of 2D symmetry protected topological order
- Gauge transformation of Matrix Product formalism
- Entanglement, localized