Integrability and Matrix Ansatz

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General context

- Thermodynamical equilibrium stationary state: no particle or energy flow.

$$P_{eq}(\mathcal{C}) \sim e^{-\frac{E(\mathcal{C})}{k_B T}}$$

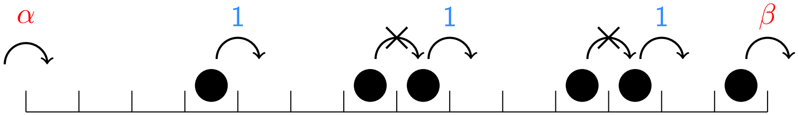
- Non-equilibrium stationary state: particle or energy currents

$$P_{eq}(\mathcal{C}) \sim \ ?$$

The matrix ansatz allows exact computations to get $P_{eq}(\mathcal{C})$.

We want to understand the matrix ansatz in the integrable system framework and generalize it to other models.

The TASEP model



Stochastic process on a one dimensional lattice with boundaries

- In the bulk, particles can jump to the right at the rate 1
- On the left boundary, particles enter at the rate α
- On the right boundary, particles leave at the rate β
- Fermi-like exclusion principle

This is an out-of-equilibrium system (there is a particle current)
The model is integrable

- Denote by $\mathcal{C} = (\tau_1, \tau_2, \ldots, \tau_L)$ a configuration of the system.

 $\longrightarrow \quad (0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1)$

- Denote by $P_t(\mathcal{C})$ the probability to be in configuration $\mathcal{C}$ at time t.

Master equation

$$\frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}' \neq \mathcal{C}} M(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}') - \sum_{\mathcal{C}' \neq \mathcal{C}} M(\mathcal{C}', \mathcal{C}) P_t(\mathcal{C})$$

- At each site, we have 2 possibilities (0 or 1) $\rightarrow \mathbb{C}^2$ "local" space
- There are L sites $\rightarrow (\mathbb{C}^2)^{\otimes L}$ total configuration space

We gather the probabilities in a vector

$$|P_t\rangle = \begin{pmatrix} P_t(\ (0,\dots,0,0,0)\ ) \\ P_t(\ (0,\dots,0,0,1)\ ) \\ P_t(\ (0,\dots,0,1,0)\ ) \\ \vdots \\ P_t(\ (1,\dots,1,1,1)\ ) \end{pmatrix} \in \quad (\mathbb{C}^2)^{\otimes L}$$

Master equation : $\dfrac{d|P_t\rangle}{dt} = M\ |P_t\rangle \ \in\ (\mathbb{C}^2)^{\otimes L}$

$$\text{Master equation :} \quad \frac{d|P_t\rangle}{dt} = M\,|P_t\rangle \;\in\; (\mathbb{C}^2)^{\otimes L}$$

The Markov matrix M can be written in terms of local jump operators

$$M = B_1 + \sum_{\ell=1}^{L-1} w_{\ell,\ell+1} + \overline{B}_L \;\in\; \big(End(\mathbb{C}^2)\big)^{\otimes L}$$

$$B = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix} \; ; \; w = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; ; \; \overline{B} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix}$$

$$End(\mathbb{C}^2) \qquad\qquad End(\mathbb{C}^2 \otimes \mathbb{C}^2) \qquad\qquad End(\mathbb{C}^2)$$

$$B_1 = \underbrace{B}_{1} \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{L-1} \quad ; \quad w_{\ell,\ell+1} = \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{\ell-1} \otimes \underbrace{w}_{\ell,\ell+1} \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{L-1-\ell}$$

Master equation : $\dfrac{d|P_t\rangle}{dt} = M\,|P_t\rangle \;\in\; (\mathbb{C}^2)^{\otimes L}$

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$$B = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix} \;;\; w = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;;\; \overline{B} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix}$$

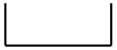
$End(\mathbb{C}^2)$ $\qquad$ $End(\mathbb{C}^2 \otimes \mathbb{C}^2)$ $\qquad$ $End(\mathbb{C}^2)$

The goal is to compute the stationary state: $M|\mathcal{S}\rangle = 0$

Matrix ansatz (Derrida, Evans, Hakim, Pasquier):

 $\longrightarrow D$

$\mathcal{S}($  $) = \dfrac{\langle\!\langle W|EDEDD|V\rangle\!\rangle}{Z_5}$

 $\longrightarrow E$

$$Z_L = \langle\!\langle W|(D+E)^L|V\rangle\!\rangle$$

Algebraic relations (DEHP)

$$DE = D + E \quad ; \quad D|V\rangle\!\rangle = \frac{1}{\beta}|V\rangle\!\rangle \quad ; \quad \langle\!\langle W|E = \frac{1}{\alpha}\langle\!\langle W|$$

DEHP gives the right weights of the stationary state:

The vector we compute using this ansatz can be written in the compact form

$$|\mathcal{S}\rangle = \frac{1}{Z_L}\langle\langle W|\, A_1\ A_2 \cdots A_L\, |V\rangle\rangle \quad \text{with} \quad A = \begin{pmatrix} E \\ D \end{pmatrix}$$

- $\langle\langle W|$ and $|V\rangle\rangle$: additional space
  They define a representation of the (E, D) algebra
- 1,2,...,L: $\mathbb{C}^2$ physical spaces $\equiv$ Sites on the lattice
- Tensor products implied:

$$A_1\ A_2 \cdots A_L = \underbrace{A \otimes A \otimes \cdots \otimes A}_{L}$$

$$A_1 = A \otimes \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{L-1} \quad \text{etc..}$$

The bulk relation $DE = D + E$ is equivalent to

$$w\ A \otimes A = \bar{A} \otimes A - A \otimes \bar{A} \quad \text{with} \quad A = \begin{pmatrix} E \\ D \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So that

$$\begin{aligned}\left(w_{12} + w_{23} + \cdots + w_{L-1,L}\right) A_1\ A_2 \cdots A_L = \\ \left(\bar{A}_1\ A_2 - A_1\ \bar{A}_2\right) A_3 \cdots A_L \\ + A_1 \left(\bar{A}_2\ A_3 - A_2\ \bar{A}_3\right) A_4 \cdots A_L \\ \vdots \\ + A_1 \cdots A_{L-2} \left(\bar{A}_{L-1}\ A_L - A_{L-1}\ \bar{A}_L\right)\end{aligned}$$

The bulk relation $DE = D + E$ is equivalent to

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So that

$$\big(w_{12} + w_{23} + \cdots + w_{L-1,L}\big)\; A_1\; A_2 \cdots A_L = \\ = \bar{A}_1\; A_2 \cdots A_L - A_1\; \cdots\; A_{L-1}\, \bar{A}_L$$

- The left boundary condition $\langle\langle W|E = \frac{1}{\alpha}\langle\langle W|$ is equivalent to

$$\langle\langle W|B\,A = -\langle\langle W|\bar{A}$$
$$\text{i.e.}\quad \langle\langle W|B_1\,A_1\;A_2\cdots A_L = -\langle\langle W|\bar{A}_1\;A_2\cdots A_L$$

- The right boundary condition $D|V\rangle\rangle = \frac{1}{\beta}|V\rangle\rangle$ is equivalent to

$$\bar{B}\,A|V\rangle\rangle = \bar{A}|V\rangle\rangle$$
$$\text{i.e.}\quad \bar{B}_L\,A_1\;A_2\cdots A_L|V\rangle\rangle = -A_2\cdots A_{L-1}\,\bar{A}_L|V\rangle\rangle$$

with $A = \begin{pmatrix} E \\ D \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

These algebraic relations lead to a telescopic sum

$$M|\mathcal{S}\rangle = \left(B_1 + \sum_{\ell=1}^{L-1} w_{\ell,\ell+1} + \overline{B}_L\right)|\mathcal{S}\rangle = 0$$

$$\begin{aligned} |\mathcal{S}\rangle &= \frac{1}{Z_L}\langle\langle W|\, A_1\ A_2 \cdots A_L\, |V\rangle\rangle \\ &= \frac{1}{Z_L}\langle\langle W|\, A \otimes A \otimes \cdots \otimes A\, |V\rangle\rangle \\ Z_L &= \langle\langle W|(D+E)^L|V\rangle\rangle \end{aligned}$$

The Matrix ansatz allows to built the stationnary state

Integrability of the TASEP

Bulk: R-matrix

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 1 & 1-x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in End(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

$x \in \mathbb{C}$ is the spectral parameter

$$P \frac{d}{dx} R(x) \Big|_{x=1} = w \quad \text{with} \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The R-matrix:

- satisfies the Yang-Baxter equation

$$R_{12}\left(\frac{x_1}{x_2}\right) R_{13}\left(\frac{x_1}{x_3}\right) R_{23}\left(\frac{x_2}{x_3}\right) = R_{23}\left(\frac{x_2}{x_3}\right) R_{13}\left(\frac{x_1}{x_3}\right) R_{12}\left(\frac{x_1}{x_2}\right)$$

$$R_{12}(x) = R(x) \otimes \mathbb{I}_2 \,, \quad R_{23}(x) = \mathbb{I}_2 \otimes R(x)$$

- is unitary $R_{12}(x)R_{21}(1/x) = 1$
- is regular $R(1) = P$
- $PR'(1) = w$
- obeys the Markovian property $\langle 1| \otimes \langle 1| R(x) = \langle 1| \otimes \langle 1|$, where $\langle 1| = (1, 1)$

Boundaries: K-matrices

$$K(x) = \begin{pmatrix} \frac{(-x\alpha+\alpha-1)x}{x\alpha-x-\alpha} & 0 \\ \frac{\alpha(x^2-1)}{x\alpha-x-\alpha} & 1 \end{pmatrix}, \quad \bar{K}(x) = \begin{pmatrix} 1 & -\frac{(x^2-1)\beta}{-x^2\beta+x\beta-x} \\ 0 & \frac{x\beta-x-\beta}{-x^2\beta+x\beta-x} \end{pmatrix}$$

$$\frac{d}{dx}K(x)\Big|_{x=1} = 2B \quad \text{and} \quad \frac{d}{dx}\bar{K}(x)\Big|_{x=1} = -2\bar{B}$$

The K-matrices:

- satisfy the reflection equation

$$R_{12}\left(\frac{x_1}{x_2}\right) K_1(x_1) R_{21}(x_1 x_2) K_2(x_2) = K_2(x_2) R_{12}(x_1 x_2) K_1(x_1) R_{21}\left(\frac{x_1}{x_2}\right)$$

- are unitarity $K(x)K(1/x) = 1$
- are regular $K(1) = 1$
- $K'(1) = 2B$ and $\bar{K}'(1) = -2\bar{B}$
- obey the Markovian property $\langle 1|K(x) = \langle 1|$

Integrability: we define the usual double row transfer matrix

$$t(x) = tr_0\Big(\widetilde{K}_0(x)\ R_{0L}(x)\cdots R_{01}(x)\ K_0(x)\ R_{10}(x)\cdots R_{L0}(x)\Big),$$

where $\widetilde{K}$ is linked to $\bar{K}$ in the following way

$$\bar{K}_1(x) = tr_0\Big(\widetilde{K}_0(\frac{1}{x})R_{01}(\frac{1}{x^2})P_{01}\Big)$$

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Properties

- $[t(x), t(x')] = 0$ (integrability of the model)
- It generates the Markov matrix

$$\frac{1}{2}\left.\frac{dt(x)}{dx}\right|_{x=1} = \frac{1}{2}K_1'(1) + \sum_{k=1}^{L-1} P_{k,k+1}R'_{k,k+1}(1) - \frac{1}{2}\bar{K}_L'(1) = M$$

- Integrability ensures that we can construct exactly the eigenvectors of the transfer matrix (and thus of the Markov matrix) using for instance the Bethe anstaz
- Bethe vectors: $t(x)\,\mathbb{B}(\bar{u}) = \lambda(x,\bar{u})\,\mathbb{B}(\bar{u})$ when the set of parameters $\bar{u}$ obeys the so-called Bethe equations
- However the stationary state is not easily obtained in this way since one should solve the Bethe equations to identify it

$\Rightarrow$ We want to recover the Matrix ansatz from R and K matrices

We want to generate the DEHP relations from R and K matrices

Let's define the vector

$$A(x) = \begin{pmatrix} E - 1 + x \\ D - 1 + \frac{1}{x} \end{pmatrix} \quad \Rightarrow \quad A(1) = \begin{pmatrix} E \\ D \end{pmatrix}, \; A'(1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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ZF relation : $\quad R_{12}\left(\frac{x_1}{x_2}\right) A_1(x_1)\, A_2(x_2) = A_2(x_2)\, A_1(x_1)$

$$\Leftrightarrow \quad DE = D + E$$

GZ relation (Left boundary): $\langle\!\langle W|A(x) = \langle\!\langle W|K(x)A\left(\frac{1}{x}\right)$

$$\Leftrightarrow \quad \langle\!\langle W|E = \frac{1}{\alpha}\langle\!\langle W|$$

GZ relation (Right boundary): $A(x)|V\rangle\!\rangle = \bar{K}(x)A\left(\frac{1}{x}\right)|V\rangle\!\rangle$

$$\Leftrightarrow \quad D|V\rangle\!\rangle = \frac{1}{\beta}|V\rangle\!\rangle$$

Integrable approach to Matrix Ansatz:

We start with a unitary R-matrix $R_{12}(x)$ of size $r^2 \times r^2$, which obeys YBE, and the corresponding unitary boundary matrices $K(x)$ and $\bar{K}(x)$, of size $r \times r$, obeying the reflection equation.

Let $A(x) = \begin{pmatrix} X_1(x) \\ X_2(x) \\ \vdots \\ X_r(x) \end{pmatrix}$ where the elements $X_1(x), \dots, X_r(x)$ belong to some algebra.

The algebraic relations for the elements $X_1(x), \dots, X_r(x)$ are provided by the following relations

Bulk exchange relations:

ZF algebra

$$R_{12}\left(\frac{x_1}{x_2}\right) A_1(x_1)A_2(x_2) = A_2(x_2)A_1(x_1)$$

- Associativity of this algebra ensured by the Yang-Baxter relation.
- Another consistency relation ensured by the unitarity of the R-matrix.

$$PR'_{12}(1)\, A_1(1)\, A_2(1) = A_1(1)\, A'_2(1) - A'_1(1)\, A_2(1)$$

N.B.: $\bar{A} = A'(1)$ is not necessarily scalar!

Conditions on the boundaries:

GZ relation

$$\langle\langle W|A(x) = \langle\langle W|K(x)A\left(\frac{1}{x}\right), \quad A(x)|V\rangle\rangle = \bar{K}(x)A\left(\frac{1}{x}\right)|V\rangle\rangle$$

- A consistency relation is ensured by the reflection equation.
- Another consistency relation is ensured by the unitarity of the K matrices.

$$\langle\langle W|\, K'(1)A(1) = \langle\langle W|2A'(1), \quad \bar{K}'(1)A(1)|V\rangle\rangle = 2A'(1)|V\rangle\rangle$$

Define the Markov matrix $M = B_1 + \sum_{\ell=1}^{L-1} w_{\ell,\ell+1} + \overline{B}_L$ with local jump operators:

- $PR'(1) = w$.
- $K'(1) = 2B$ and $\bar{K}'(1) = -2\bar{B}$.

Then the vector

$$|\mathcal{S}\rangle = \langle\!\langle W|A_1(1)A_2(1)\cdots A_L(1)|V\rangle\!\rangle,$$

is the stationary state of the process (we have again a telescopic sum):

$$M|\mathcal{S}\rangle = 0$$

More generally, we define the vector

$$|\mathcal{S}(\theta_1,\dots,\theta_L)\rangle = \langle\!\langle W|A_1(\theta_1)\cdots A_L(\theta_L)|V\rangle\!\rangle$$

Using the ZF and GZ relations one shows

$$\begin{aligned}|\mathcal{S}(\theta_1,\dots,\theta_L)\rangle &= t(\theta_i|\bar{\theta})\;|\mathcal{S}(\theta_1,\dots,\theta_L)\rangle,\\ |\mathcal{S}(\theta_1,\dots,\theta_L)\rangle &= t(1/\theta_i|\bar{\theta})\;|\mathcal{S}(\theta_1,\dots,\theta_L)\rangle,\end{aligned}$$

with the inhomogeneous transfer matrix

$$t(x|\bar{\theta}) = tr_0\Big(\;\widetilde{K}_0(x)\;R_{0L}(\frac{x}{\theta_L})\cdots R_{01}(\frac{x}{\theta_1})\;K_0(x)\;R_{10}(x\theta_1)\cdots R_{L0}(x\theta_L)\;\Big)$$

With the crossing symmetry of the R and K matrices:

$$t(x|\bar{\theta}) = \Big(\lambda(x|\bar{\theta}) - 1\Big)\ t(1/xq|\bar{\theta})$$

$\lambda(x|\bar{\theta})$ depends on the considered model.

We get through an interpolation in x

$$t(x|\bar{\theta})\ |\mathcal{S}(\theta_1,\dots,\theta_L)\rangle = \lambda(x|\bar{\theta})|\mathcal{S}(\theta_1,\dots,\theta_L)\rangle$$

Summary

Starting with an R-matrix and two K boundary matrices:

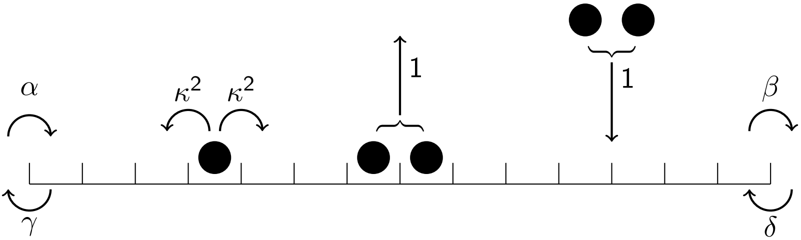
- **We can construct an integrable model**
  If R obeys the Yang-Baxter eq. and K, $\widetilde{K}$ obey the reflection eq.
- **We can construct the steady state of the model**
  Using a ZF algebra and GZ relations to define the Matrix ansatz

  The steady state is an eigenvector of the transfer matrix $t(x|\bar{\theta})$

NB: Contrarily to the Bethe ansatz there is no Bethe parameters (and no Bethe eqs). However, we get only one state...

Example: a reaction-diffusion model

Description of the model:



$$M = B_1 + \sum_{\ell=1}^{L-1} w_{\ell,\ell+1} + \overline{B}_L$$

$$B = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \; w = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -\kappa^2 & \kappa^2 & 0 \\ 0 & \kappa^2 & -\kappa^2 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \; \overline{B} = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$

R-matrix

$$R(x) = \begin{pmatrix} \frac{\kappa(x+1)}{\kappa(x+1)+x-1} & 0 & 0 & \frac{x-1}{\kappa(x+1)+x-1} \\ 0 & \frac{\kappa(x-1)}{\kappa(x-1)+x+1} & \frac{x+1}{\kappa(x-1)+x+1} & 0 \\ 0 & \frac{x+1}{\kappa(x-1)+x+1} & \frac{\kappa(x-1)}{\kappa(x-1)+x+1} & 0 \\ \frac{x-1}{\kappa(x+1)+x-1} & 0 & 0 & \frac{\kappa(x+1)}{\kappa(x+1)+x-1} \end{pmatrix}$$

- Satisfies Yang-Baxter equation, unitarity and regularity relations
- Generates the bulk's local jump operator: $PR'(1) = \frac{1}{2\kappa}\, w$

K matrices

$$K(x) = \begin{pmatrix} \frac{(x^2+1)((x^2-1)(\gamma-\alpha)+4x\kappa)}{2x((x^2-1)(\alpha+\gamma)+2\kappa(x^2+1))} & \frac{(x^2-1)((x^2+1)(\gamma-\alpha)+2x(\alpha+\gamma))}{2x((x^2-1)(\alpha+\gamma)+2\kappa(x^2+1))} \\ -\frac{(x^2-1)((x^2+1)(\gamma-\alpha)-2x(\alpha+\gamma))}{2x((x^2-1)(\alpha+\gamma)+2\kappa(x^2+1))} & -\frac{(x^2+1)((x^2-1)(\gamma-\alpha)-4x\kappa)}{2x((x^2-1)(\alpha+\gamma)+2\kappa(x^2+1))} \end{pmatrix}$$

$$\bar{K}(x) = \begin{pmatrix} \frac{(x^2+1)((x^2-1)(\delta-\beta)+4x\kappa)}{2x(-(x^2-1)(\delta+\beta)+2\kappa(x^2+1))} & \frac{(x^2-1)((x^2+1)(\delta-\beta)-2x(\delta+\beta))}{2x(-(x^2-1)(\delta+\beta)+2\kappa(x^2+1))} \\ -\frac{(x^2-1)((x^2+1)(\delta-\beta)+2x(\delta+\beta))}{2x(-(x^2-1)(\delta+\beta)+2\kappa(x^2+1))} & -\frac{(x^2+1)((x^2-1)(\delta-\beta)-4x\kappa)}{2x(-(x^2-1)(\delta+\beta)+2\kappa(x^2+1))} \end{pmatrix}$$

- Satisfy the reflection equation, the unitarity and regularity relations
- Generate the boundaries local jump operators: $K'(1) = \frac{1}{\kappa}\, B$ and $\bar{K}'(1) = -\frac{1}{\kappa}\, \bar{B}$

Construction of the matrix ansatz algebra:

The following expansion of the vector $A(x)$ turns out to be optimal

$$A(x) = \begin{pmatrix} G_1 x + G_2 + \frac{G_3}{x} \\ -G_1 x + G_2 - \frac{G_3}{x} \end{pmatrix}$$

Construction of the matrix ansatz algebra:

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Exchange relations in the bulk (ZF algebra):

$$R_{12}\left(\frac{x_1}{x_2}\right) A_1(x_1)\, A_2(x_2) = A_2(x_2)\, A_1(x_1) \;\Leftrightarrow\; \begin{cases} \phi\; G_1 G_2 = G_2 G_1 \\ G_1 G_3 = G_3 G_1 \\ \phi\; G_2 G_3 = G_3 G_2 \end{cases}$$

with $\phi = \dfrac{\kappa - 1}{\kappa + 1}$

Condition on the left boundary (GZ relation):

$$\langle\!\langle W|A(x) = \langle\!\langle W|K(x)A\left(\frac{1}{x}\right) \;\Leftrightarrow\; \langle\!\langle W|(G_1 - c\,G_2 - a\,G_3) = 0$$

with $a = \dfrac{2\kappa - \alpha - \gamma}{2\kappa + \alpha + \gamma}$ and $c = \dfrac{\gamma - \alpha}{2\kappa + \alpha + \gamma}$

Condition on the right boundary (GZ relation):

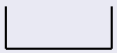
$$A(x)|V\rangle\!\rangle = \bar{K}(x)A\left(\frac{1}{x}\right)|V\rangle\!\rangle \;\Leftrightarrow\; (G_3 - b\,G_1 - d\,G_2)|V\rangle\!\rangle = 0$$

with $b = \dfrac{2\kappa - \delta - \beta}{2\kappa + \delta + \beta}$ and $d = \dfrac{\beta - \delta}{2\kappa + \delta + \beta}$

Matrix ansatz:

$$A(1) = \begin{pmatrix} G_2 + (G_1 + G_3) \\ G_2 - (G_1 + G_3) \end{pmatrix} \quad \text{and} \quad A'(1) = (G_1 - G_3) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\longrightarrow \; D = G_2 - G_1 - G_3$$



$$\longrightarrow \; E = G_2 + G_1 + G_3$$

Normalization factor:

$$Z_L = \langle\langle W | (D+E)^L | V \rangle\rangle = 2^L \langle\langle W | G_2^L | V \rangle\rangle$$

Computation of the densities:

$$\langle n_i \rangle = \frac{\langle\langle W|(D+E)^{i-1}D(D+E)^{L-i}|V\rangle\rangle}{\langle\langle W|(D+E)^{L}|V\rangle\rangle} = \frac{1}{2}\frac{\langle\langle W|G_2^{i-1}(G_2-G_1-G_3)G_2^{L-i}|V\rangle\rangle}{\langle\langle W|G_2^{L}|V\rangle\rangle}$$

Using solely

$$\begin{cases} \phi\ G_1 G_2 = G_2 G_1 \\ G_1 G_3 = G_3 G_1 \\ \phi\ G_2 G_3 = G_3 G_2 \end{cases} \quad \text{and} \quad \begin{cases} \langle\langle W|(G_1 - cG_2 - aG_3) = 0 \\ (G_3 - bG_1 - dG_2)|V\rangle\rangle = 0 \end{cases}$$

we get $\langle n_i \rangle = \frac{1}{2} - \frac{c\phi^{i-1} + ad\phi^{L+i-2} + d\phi^{L-i} + bc\phi^{2L-i-1}}{2(1-ab\phi^{2L-2})}$

We found a representation where $Z_L = \langle\langle W|G_2^L|V\rangle\rangle$ is non-zero

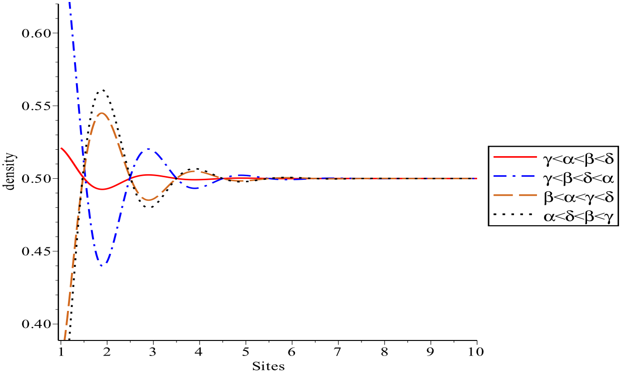


Figure: Density profiles close to the left boundary for various values of the boundary parameters

We can also compute the currents.

Diffusion current from the site i to $i+1$:

$$\langle J^{lat}_{i\to i+1}\rangle = \frac{\kappa^2}{\kappa+1}\;\frac{d\phi^{L-i-1}+bc\phi^{2L-i-2}-c\phi^{i-1}-ad\phi^{L+i-2}}{1-ab\phi^{2L-2}}$$

Evaporation current at sites $(i,i+1)$:

$$\langle J^{eva}_{i,i+1}\rangle = -\frac{\kappa}{\kappa+1}\;\frac{c\phi^{i-1}+ad\phi^{L+i-2}+d\phi^{L-i-1}+bc\phi^{2L-i-2}}{1-ab\phi^{2L-2}}$$

Conclusion

- Comprehension of the matrix ansatz in the integrable systems framework
- 2 key ingredients: the ZF algebra (in the bulk) and the GZ relations (on the boundaries)
- In principle it allows us to construct a matrix ansatz for **any** integrable reaction-diffusion process (if we know the R-matrix)
- In practice the matrix ansatz algebra can be very complicated and the computation of observables can be very hard

Perspectives

- Make a better use of the spectral parameter (with the ZF and GZ relations) to extract property of the stationary state
- relations with orthogonal polynomials (see e.g. arXiv:1506.04874 for shifted Schur polynomials in relation with discrete time TASEP)
- More efficient computation of observables
- Find a prescription for the truncation in the vector $A(x)$ (optimal number of generators)
- Use this framework to compute cumulants (work in progress for the example)
- Solve more complicated models: for instance we solved a 2-species TASEP with boundaries (see arXiv:1412.5939 with K. Mallick)

Thank you!

Representation for the algebra of the reaction-diffusion model

$$\phi\ G_1 G_2 = G_2 G_1\ ;\quad G_1 G_3 = G_3 G_1\ ;\quad \phi\ G_2 G_3 = G_3 G_2$$

$$G_1 = g_1 \otimes 1, \quad G_2 = g_2 \otimes g_2, \quad G_3 = 1 \otimes g_3,$$

where

$$g_1 = \sum_{n=0}^{+\infty} |n+1\rangle\!\rangle \langle\!\langle n| \quad ; \quad g_2 = \sum_{n=0}^{+\infty} \phi^n |n\rangle\!\rangle \langle\!\langle n| \quad ; \quad g_3 = \sum_{n=1}^{+\infty} |n-1\rangle\!\rangle \langle\!\langle n|$$

We have used $\{|n\rangle\!\rangle \mid n \geq 0\}$ as an infinite basis of the additional space.

$$\begin{cases} \langle\!\langle W|\big(G_1 - cG_2 - aG_3\big) = 0 \\ \big(G_3 - bG_1 - dG_2\big)|V\rangle\!\rangle = 0 \end{cases}$$

$$|V\rangle\!\rangle = \sum_{n,m=0}^{+\infty} v_{n,m}\,|n\rangle\!\rangle \otimes |m\rangle\!\rangle \quad ; \quad v_{n,m} = d^{m-n} b^n \frac{\phi^{\frac{(m-n)(m-n-1)}{2}}}{(1-\phi^2)\cdots(1-\phi^{2n})}$$

$$\langle\!\langle W| = \sum_{n,m=0}^{+\infty} w_{n,m}\,\langle\!\langle n| \otimes \langle\!\langle m| \quad ; \quad w_{n,m} = c^{n-m} a^m \frac{\phi^{\frac{(n-m)(n-m-1)}{2}}}{(1-\phi^2)\cdots(1-\phi^{2m})}$$

$$
\begin{aligned}
Z_L &= \langle\!\langle W | G_2^L | V \rangle\!\rangle \\
&= \sum_{n,m=0}^{+\infty} \frac{\left(\phi^L\, cb/d\right)^n\, \left(\phi^L\, da/c\right)^m\, \phi^{(m-n)^2}}{(1-\phi^2)\cdots(1-\phi^{2n}) \times (1-\phi^2)\cdots(1-\phi^{2m})}
\end{aligned}
$$

The series is convergent when

$$
|\phi| < 1\,, \quad \left|\frac{b\,c\,\phi^L}{(1-\phi^2)d}\right| < 1 \quad \text{and} \quad \left|\frac{a\,d\,\phi^L}{(1-\phi^2)c}\right| < 1
$$

Thanks to the symmetry $\kappa \to -\kappa$, we can choose $|\phi| < 1$.
Then, when L is large enough, the conditions are always fulfilled (when $c, d, \kappa \neq 0$).