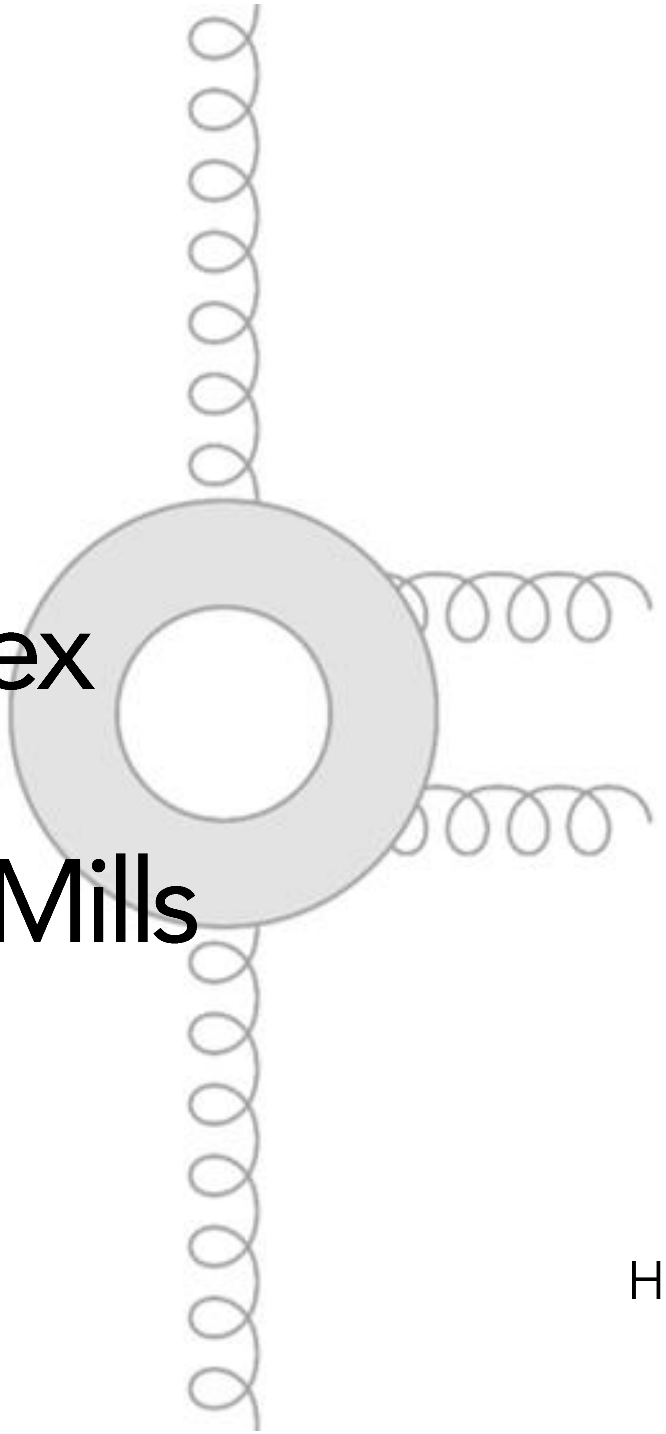
The one-loop central emission vertex for two gluons in $\mathcal{N} = 4$ super Yang-Mills



Based on forthcoming work with Vittorio Del Duca, Lance Dixon, Einan Gardi and Jenni Smillie

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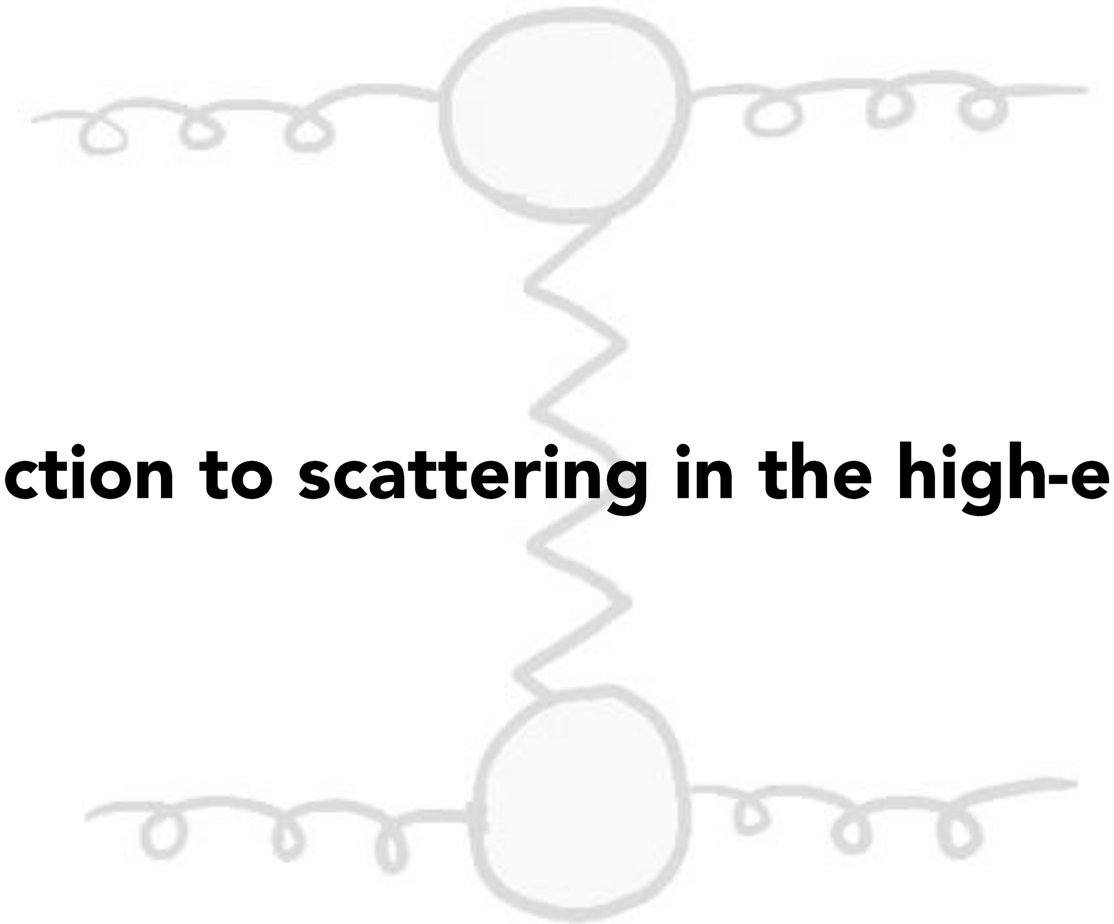
Higgs Centre Amplitudes Meeting
30th March 2022

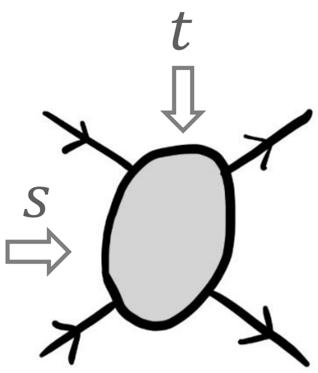


1. Introduction to scattering in the high-energy limit

High-energy logarithms

At each order in perturbative QCD, large logarithms arise when the centre of mass energy is much greater than the transverse momenta of the produced partons.

For 2 → 2 scattering we can write the cross section as



$L \equiv \log\left(\frac{s}{-t}\right) \gg 1$

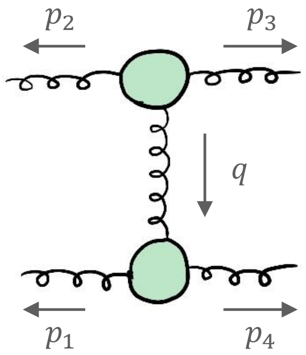
$\alpha_s \ll 1$

$\alpha_s L \sim 1$

	LL		NLL		NNLL	
$\sigma^{(0)}/\sigma^{(0)} =$	1					
$\sigma^{(1)}/\sigma^{(0)} =$	$\alpha_s L c_0^{(1)}$	$+$	$\alpha_s c_1^{(1)}$			
$\sigma^{(2)}/\sigma^{(0)} =$	$\alpha_s^2 L^2 c_0^{(2)}$	$+$	$\alpha_s^2 L\, c_1^{(2)}$	$+$	$\alpha_s^2 c_2^{(2)}$	
$\sigma^{(3)}/\sigma^{(0)} =$	$\alpha_s^3 L^3 c_0^{(3)}$	$+$	$\alpha_s^3 L^2 c_1^{(3)}$	$+$	$\alpha_s^3 L c_2^{(3)}$	$+ \ldots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	

Regge limit of the 2→2 amplitude at tree level

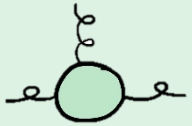
In the Regge limit the LO amplitude takes a simple factorised form:



$$\mathcal{M}^{(0)}_{gg\to gg} \xrightarrow[s\gg -t]{} 2s\left[g_s f^{a_3 c a_2} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2})\right]\left(\frac{1}{|q_\perp|^2}\right)\left[g_s f^{a_1 c a_4} C^{g(0)}(p_1^{\nu_1}, p_4^{-\nu_1})\right]$$

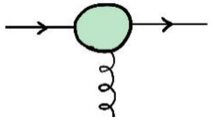
where the so-called impact factors are simple helicity conserving phases [1]:

$$C^{g(0)}(p_2^{\ominus}, p_3^{\oplus}) = 1$$



$$C^{g(0)}(p_1^{\ominus}, p_4^{\oplus}) = -\frac{p_{4\perp}^*}{p_{4\perp}}$$

Quark scattering is similarly described, with the replacement of $f^{a_i c a_j} \to T^c_{a_i a_j}$



In all cases ($gg \to gg, qg \to qg, qQ \to qQ$ etc.) the scattering is described by antisymmetric octet exchange $\mathbf{8}_a$ in the t-channel

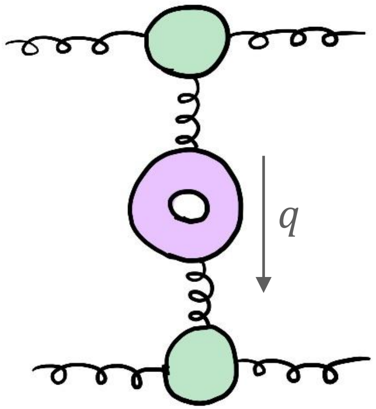
[1] hep-ph/9503340 V. Del Duca

Regge limit of the 2→2 amplitude at one loop

At loop level we might have expected other representations to be exchanged in the t-channel:

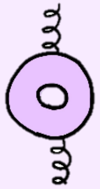
$$\mathbf{8}_a \otimes \mathbf{8}_a = \mathbf{1} \oplus \mathbf{8}_a \oplus \mathbf{8}_s \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}$$

However, in the Regge limit, the one-loop correction to the amplitude does not alter the colour structure:



$$\mathcal{M}^{(1)}_{gg\to gg} \xrightarrow[s\gg -t]{} 2s \left[g_s f^{a_3 c a_2} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \right] \times \left(\frac{1}{|q_\perp|^2} \right) \frac{\alpha_s}{4\pi} \alpha^{(1)}(q_\perp) \log\left(\frac{s}{|q_\perp|^2} \right) \times \left[g_s f^{a_1 c a_4} C^{g(0)}(p_1^{\nu_1}, p_4^{-\nu_1}) \right]$$

The LL contribution from the loop integration appears with the transverse function known as the Regge trajectory [2]:



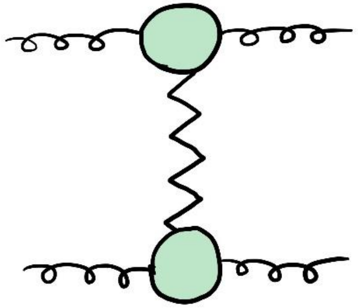
$$\alpha^{(1)}(q_\perp) = \kappa_\Gamma N_c \frac{2}{\epsilon} \left(\frac{\mu^2}{|q_\perp|^2} \right)^{\epsilon}$$

$$\kappa_\Gamma = (4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

[2] Sov. J. Nucl. Phys. 23 (1976) , Lipatov

LL behaviour of 2→2 amplitudes at all orders

To all-orders, it is found that the virtual corrections exponentiate [3]:



$$\mathcal{M}^{\rm LL}_{gg\to gg} = 2s\left[g_s f^{a_3 c a_2} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2})\right] \frac{1}{|q_\perp|^2}\left(\frac{s}{\tau}\right)^{\frac{\alpha_s}{4\pi}\alpha^{(1)}(q_\perp)} \left[g_s f^{a_1 c a_4} C^{g(0)}(p_1^{\nu_1}, p_4^{-\nu_1})\right]$$

Here $\tau > 0$ is some high-energy factorisation scale of the same order as $|q_\perp|^2$.

This exponential can be seen to be a modification of the gluon propagator, known as gluon Reggeisation:

$$\frac{1}{|q_\perp|^2} \to \frac{1}{|q_\perp|^2}\left(\frac{s}{\tau}\right)^{\frac{\alpha_s}{4\pi}\alpha^{(1)}(q_\perp)}$$

In order to find the LL behaviour of the cross section we must also study real emissions.

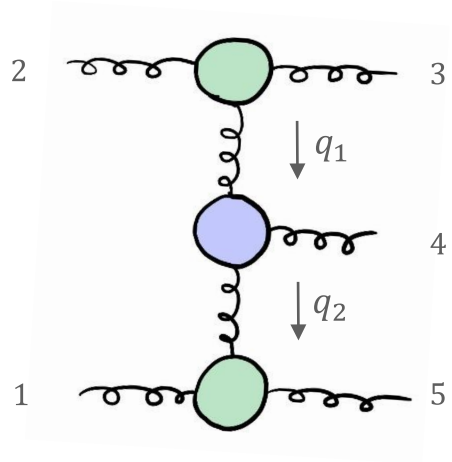
[3] Sov. Phys. JETP, Vol.44, No. 3 (1976) Kuraev, Lipatov, Fadin

2→3 amplitude at tree-level

The LL contribution from the phase-space integration of 2 → 3 real radiative corrections comes from the Multi-Regge Kinematic (MRK) region:

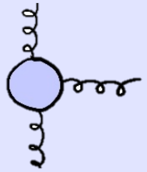
$$y_3 \gg y_4 \gg y_5, \qquad |p_{3\perp}| \approx |p_{4\perp}| \approx |p_{5\perp}|$$

At LO the amplitude again has a simple factorised form:



$$\mathcal{M}^{(0)}_{gg\to ggg} \xrightarrow[\text{MRK}]{} 2s_{12}\left[g_s f^{a_3 c a_2} C^{g(0)}(p_2^{\nu_2}, p_3^{\nu_3})\right]\left(\frac{1}{|q_{1\perp}|^2}\right)$$
$$\times \left[g_s f^{c_1 c_2 a_4} V^{g(0)}(q_1, p_4^{\nu_4}, q_2)\right]$$
$$\times \left(\frac{1}{|q_{2\perp}|^2}\right)\left[g_s f^{a_1 c a_5} C^{g(0)}(p_1^{\nu_2}, p_5^{\nu_5})\right]$$

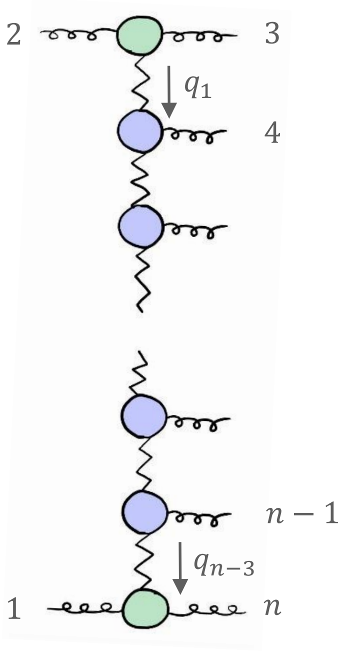
Where the contribution of all gluon emissions is given by the effective Lipatov vertex [2]:



$$V^{g(0)}(q_1, p^{\oplus}, q_2) = \frac{q_{1\perp}^* q_{2\perp}}{p_\perp}$$

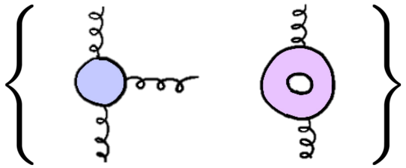
QCD at LL

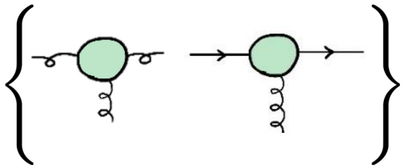
The $2 \to n$ amplitude in QCD is described to all orders by [3]



$$
\begin{aligned}
\mathcal{M}^{\mathrm{LL}}_{gg\to(n-2)g} = 2s_{12} & \left[g_s f^{a_3 c_1 a_2} C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2})\right] \\
& \times \prod_{i=4}^{n-1} \left[g_s f^{c_{i-3} c_{i-2} a_i} V^{g(0)}(q_{i-3}, p_i^{\nu_i}, q_{i-2})\right] \\
& \times \prod_{i=1}^{n-3} \frac{1}{|q_{i\perp}|^2} \left(\frac{s_{i+2,i+3}}{\tau}\right)^{\frac{\alpha_s}{4\pi}\alpha^{(1)}(q_{i\perp})} \\
& \times \left[g_s f^{a_1 c a_n} C^{g(0)}(p_1^{\nu_1}, p_n^{-\nu_1})\right]
\end{aligned}
$$

This was the form from which the BFKL equation was first derived [4], which describes the evolution of a t-channel gluon ladder in rapidity.

The kernel of this equation is obtained from the simple ingredients: 

Cross sections can be computed with this framework using the impact factors: 

[4] Sov. J. Nucl. Phys. 28 (1978) Balitsky, Lipatov

Signature

We can consider 2 → 2 amplitudes with definite crossing under $s \leftrightarrow u$. In the Regge limit we have $s \approx -u$ so this becomes crossing under $s \leftrightarrow -s$:

$$\mathcal{M}_4^{(\pm)}(s,t) = \frac{1}{2}\left(\mathcal{M}_4(s,t) \pm \mathcal{M}_4(-s,t)\right)$$

In ref. [5] it was shown that the 'odd' (-) part of the amplitude is real while the 'even' (+) part of the amplitude is imaginary.

To LL accuracy we have seen:

$$\left.\mathcal{M}_4^{(-)}\right|_{\mathrm{LL}} = \left.\mathcal{M}_4^{[\mathbf{8}_a]}\right|_{\mathrm{LL}}, \qquad \left.\mathcal{M}_4^{(+)}\right|_{\mathrm{LL}} = 0$$

At NLL there is a nonzero contribution from the even amplitude which is not described by the simple Regge scaling that we have seen so far.

However, for the odd amplitude the LL results can be generalised.

[5] arXiv:1701.05241 Caron-Huot, Gardi, Vernazza,

2→2 amplitude at NLL

The real part of the amplitude is given by

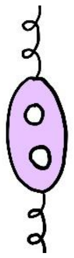
$$\mathrm{Re}\left[\mathcal{M}^{\mathrm{NLL}}_{gg\to gg}\right] = 2s\, C^g(p_2^{\nu_2}, p_3^{\nu_3}) \frac{1}{t}\frac{1}{2}\left[\left(\frac{s}{\tau}\right)^{\alpha(t)} + \left(\frac{-s}{\tau}\right)^{\alpha(t)}\right] C^g(p_1^{\nu_1}, p_4^{\nu_4})$$

where we have introduced the all-orders definitions

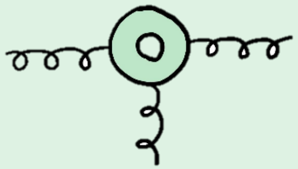
$$C^g(p_2^{\nu_2}, p_3^{\nu_3}) = g_s f\left(C^{g(0)}(p_2^{\nu_2}, p_3^{\nu_3}) + \left(\frac{\alpha_s}{4\pi}\right) C^{g(1)}(p_2^{\nu_2}, p_3^{\nu_3}) + \mathcal{O}(\alpha_s^2)\right)$$

$$\alpha(t) = \frac{\alpha_s}{4\pi}\alpha^{(1)}(t) + \left(\frac{\alpha_s}{4\pi}\right)^2 \alpha^{(2)}(t)$$

As we are considering contributions of the order $\alpha_s(\alpha_s L)$ we need to know the two-loop Regge trajectory [6]:



and the one-loop impact factors, e.g. [7]:



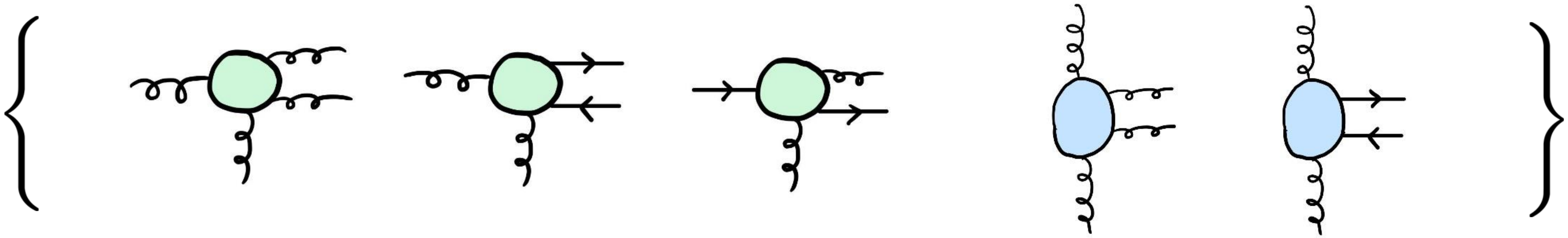
$$C^{g(1)}(p_2^{\nu_2}, p_3^{\nu_3}) = N_c \kappa_\Gamma C^{g(0)}(p_2^{\nu_2}, p_3^{\nu_3}) \left(\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\log\left(\frac{\tau}{-t}\right) + \frac{\pi^2}{2}\right) + \cdots\right)$$

[6] Phys. Lett. B 359 (1995) Fadin, Kotsky, Fiore
[7] Nucl. Phys. B 406 (1993) Fadin, Lipatov,

QCD at NLL

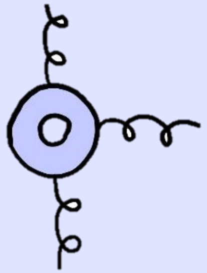
We must also consider the contribution of the integrated real corrections. The NLL contribution is obtained by relaxing one of the strong rapidity orderings, known as Next-to-Multi-Regge Kinematics (NMRK).

We therefore need the LO factorised expressions for 2-parton emission vertices



As well as the one-loop correction to the Lipatov vertex [7,8]

$$V^g(q_1, p_4^{\nu_4}, q_2) = i g_s f^{c_1,c_2,a_4} \left(V^{g(0)}(q_1, p_4^{\nu_4}, q_2) + \left(\frac{\alpha_s}{4\pi}\right) V^{g(1)}(q_1, p_4^{\nu_4}, q_2) + \mathcal{O}(\alpha_s^2) \right)$$

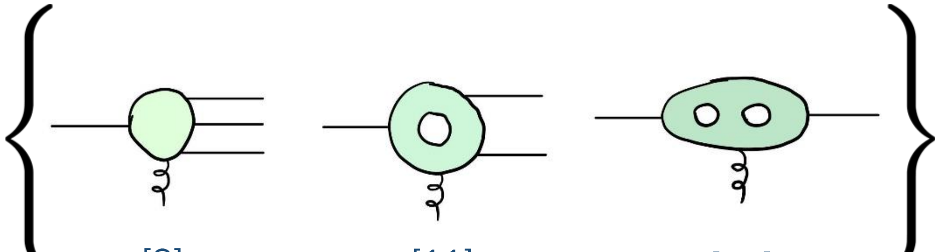


$$V^{g(1)}(q_1, p_4^{\nu_4}, q_2) = N_c \kappa_\Gamma V^{g(0)}(q_1, p_4^{\nu_4}, q_2) \Bigg(-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{|p_{4\perp}|^2}\right)^{\epsilon} + \frac{1}{\epsilon}\left[\left(\frac{\mu^2}{-t_1}\right)^{\epsilon} + \left(\frac{\mu^2}{-t_2}\right)^{\epsilon}\right] \log\left(\frac{\tau}{|p_{4\perp}|^2}\right) - \frac{1}{2}\log^2\left(\frac{t_1}{t_2}\right) + \frac{\pi^2}{3} + \cdots \Bigg)$$

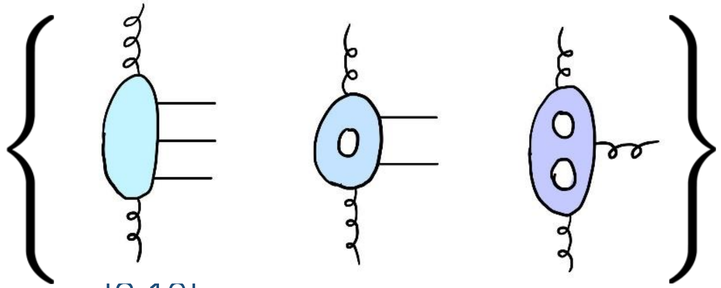
[8] hep-ph/9810215 Del Duca, Schmidt

QCD at NNLL

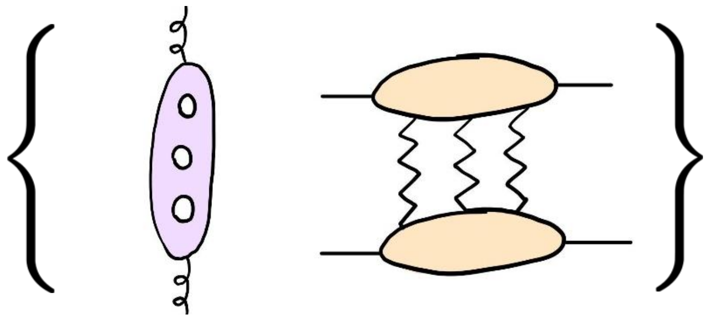
Finally, we give an overview of the factorised expressions which are necessary to describe QCD to Next-to-Next-to-Leading Logarithmic accuracy:



[9] [11] [12]



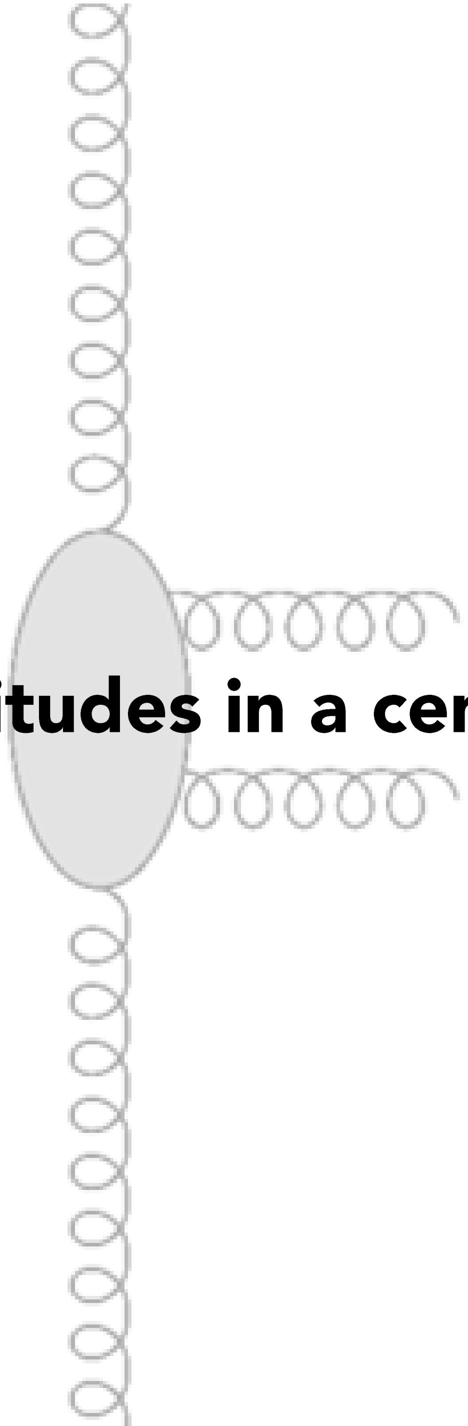
[9,10]



[13] [14]

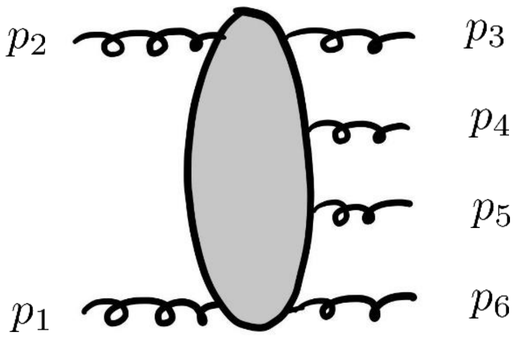
As for the imaginary part of the amplitude at NLL, the real part of the amplitude will also contain a contribution from a Regge cut, due to 3-Reggeon exchange.

[9] hep-ph/9909464 Del Duca, Frizzo, Maltoni; [10] hep-ph/0411185 Antonov, Lipatov, Kuraev, Cherednikov, [11] arXiv:2103.16593 Canay, Del Duca;
[12] arXiv:1409.8330 Del Duca, Falcioni, Magnea, Vernazza; [13] arXiv:2111.14265 Del Duca, Marzucca, Verbeek
[14] arXiv:2112.11098 Falcioni, Gardi, Maher, Milloy, Vernazza;



2. Tree-level gluon amplitudes in a central NMRK limit

6 parton kinematics

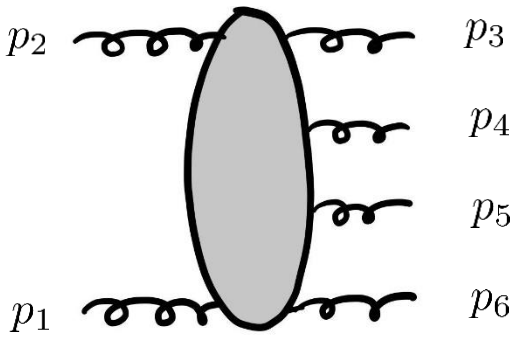


In terms of the standard representation of momenta, $p^\mu = (p^0, p^x, p^y, p^z)$
we use the lightcone momenta:

$$p^\pm = p^0 \pm p^z \qquad\qquad p_\perp = p^x + i p^y$$

General Kinematics	Central NMRK	MRK
$p_3^+ \sim p_4^+ \sim p_5^+ \sim p_6^+$	$p_3^+ \gg p_4^+ \sim p_5^+ \gg p_6^+$	$p_3^+ \gg p_4^+ \gg p_5^+ \gg p_6^+$
$p_3^- \sim p_4^- \sim p_5^- \sim p_6^-$	$p_3^- \ll p_4^- \sim p_5^- \ll p_6^-$	$p_3^- \ll p_4^- \ll p_5^- \ll p_6^-$
s_{34}	$p_3^+ p_4^-$	$p_3^+ p_4^-$
s_{45}	$p_4^+ p_5^- + p_5^+ p_4^- - p_{4\perp}^* p_{5\perp} - p_{5\perp}^* p_{4\perp}$	$p_4^+ p_5^-$
s_{56}	$p_5^+ p_6^-$	$p_5^+ p_6^-$

6 parton kinematics



In terms of the standard representation of momenta, $p^\mu = (p^0, p^x, p^y, p^z)$ we use the lightcone momenta:

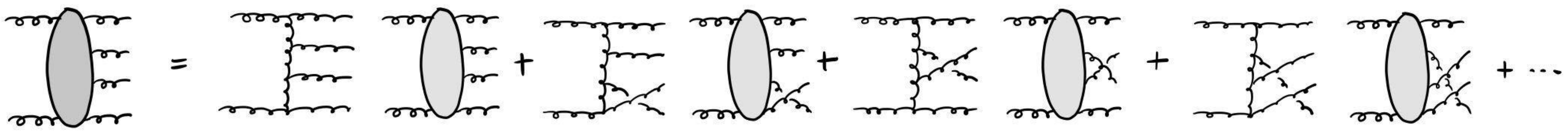
$$p^\pm = p^0 \pm p^z \qquad p_\perp = p^x + ip^y$$

General Kinematics	Central NMRK	MRK
$p_3^+ \sim p_4^+ \sim p_5^+ \sim p_6^+$	$p_3^+ \gg p_4^+ \sim p_5^+ \gg p_6^+$	$p_3^+ \gg p_4^+ \gg p_5^+ \gg p_6^+$
$p_3^- \sim p_4^- \sim p_5^- \sim p_6^-$	$p_3^- \ll p_4^- \sim p_5^- \ll p_6^-$	$p_3^- \ll p_4^- \ll p_5^- \ll p_6^-$
$\langle 3\ 4 \rangle$	$-p_{4\perp}\sqrt{\frac{p_3^+}{p_4^+}}$	$-p_{4\perp}\sqrt{\frac{p_3^+}{p_4^+}}$
$\langle 4\ 5 \rangle$	$p_{4\perp}\sqrt{\frac{p_5^+}{p_4^+}} - p_{5\perp}\sqrt{\frac{p_4^+}{p_5^+}}$	$-p_{5\perp}\sqrt{\frac{p_4^+}{p_5^+}}$
$\langle 5\ 6 \rangle$	$-p_{6\perp}\sqrt{\frac{p_5^+}{p_6^+}}$	$-p_{6\perp}\sqrt{\frac{p_5^+}{p_6^+}}$

Tree-level six gluon amplitude

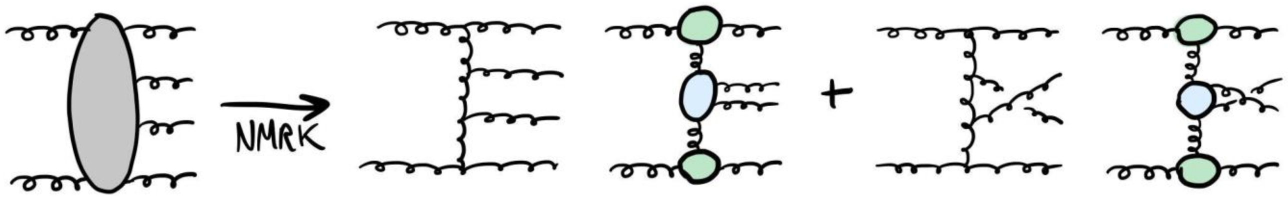
In the colour basis introduced in ref. [15], the tree-level six-gluon amplitude in general kinematics is given by a sum over 4! partial amplitudes:

$$\mathcal{M}_{6g}^{(0)} = g_s^4 \sum_{\sigma\in S_4} (F^{a_{\sigma_3}} F^{a_{\sigma_4}} F^{a_{\sigma_5}} F^{a_{\sigma_6}})_{a_2 a_1} M_{6g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_{\sigma_3}^{\nu_{\sigma_3}}, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, p_{\sigma_6}^{\nu_{\sigma_6}})$$



In the central NMRK limit, all but two of these kinematic terms are suppressed:

$$\mathcal{M}_{6g}^{(0)} \xrightarrow[\text{NMRK}]{} = g_s^4 \sum_{\sigma\in S_2} (F^{a_3})_{a_2 c_1} (F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3} (F^{a_6})_{a_1 c_3} M_{6g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{-\nu_2}, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, p_6^{-\nu_1})$$

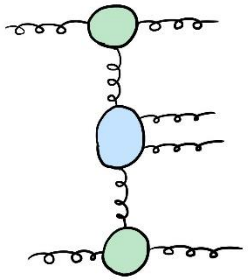


We call these orderings $\sigma_A = \{1,2,3,4,5,6\}$ and $\sigma_{A'} = \{1,2,3,5,4,6\}$.

[15] hep-th/0501052 Del Duca, Dixon, Maltoni

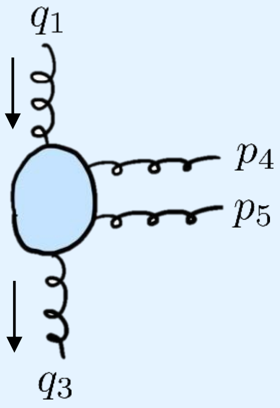
Tree-level two-gluon CEV

For the σ_A ordering, the partial amplitude factorises:



$$M_{6g}^{(0)}(p_1^{\nu_1}, p_2^{\nu_2}, p_3^{\nu_3}, p_4^{\nu_4}, p_5^{\nu_5}, p_6^{\nu_6}) \xrightarrow[\text{NMRK}]{} s\, C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2}) \frac{1}{t_1} A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}),$$

The central physics is described by the two-gluon central-emission vertex [16]:



$$A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) = \frac{q_{1\perp}^* q_{3\perp}}{p_{4\perp}} \sqrt{\frac{p_4^+}{p_5^+}} \frac{1}{\langle 45 \rangle}$$

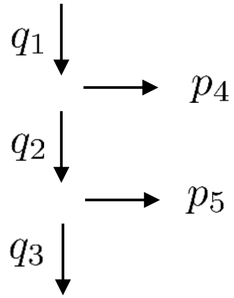
$$A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) = -\frac{p_{4\perp}^*}{p_{4\perp}} \Bigg\{ -\frac{1}{s_{45}} \left[\frac{y_5\, p_{5\perp} |q_{1\perp}|^2}{p_{5\perp}^*} + \frac{x_4\, p_{4\perp} |q_{3\perp}|^2}{p_{4\perp}^*} + \frac{s_{234}\, p_{4\perp} p_{5\perp}}{p_4^- p_5^+} \right] + \frac{(q_{3\perp} + p_{5\perp})^2}{s_{234}} - \frac{q_{3\perp} + p_{5\perp}}{s_{45}} \left[\frac{p_{4\perp}}{y_4} - \frac{p_{5\perp}}{x_5} \right] \Bigg\},$$

The remaining colour ordering $\sigma_{A'}$ is related by the simple exchange of $p_4^{\nu_4} \leftrightarrow p_5^{\nu_5}$

[16] JETP Lett. 49 (1989) Fadin, Lipatov; hep-ph/9601211 Del Duca

Before we leave tree level 1: Minimal Variables

To simplify the manipulation of the rational terms we will encounter at one-loop, let's introduce a set of minimal variables (2+3 real parameters) to describe two emissions in a central NMRK limit:



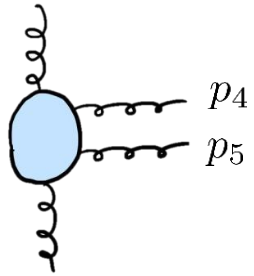
$$p_{4\perp} = q_{1\perp} - q_{2\perp} \qquad 1 = \frac{p_{4\perp}}{q_{1\perp}} + \frac{q_{2\perp}}{q_{1\perp}} = \frac{1}{1-z} + \frac{-z}{1-z} \qquad z = -\frac{q_{2\perp}}{p_{4\perp}}$$

$$p_{5\perp} = q_{2\perp} - q_{3\perp} \qquad 1 = \frac{q_{2\perp}}{q_{3\perp}} - \frac{p_{5\perp}}{q_{3\perp}} = \frac{-w}{1-w} - \frac{-1}{1-w} \qquad w = \frac{q_{2\perp}}{p_{5\perp}}$$

Some Lorentz invariant quantities in the NMRK limit are:

$$\langle 4\ 5 \rangle = -q_{1\perp} \frac{w + Xz}{w\sqrt{X}(z-1)}, \quad s_{123} = |q_{1\perp}|^2 \frac{p_4^+}{p_6^+} \frac{(1+X)|w-1|^2|z|^2}{X|z-1|^2|w|^2}, \quad s_{234} = -|q_{1\perp}|^2 \frac{(1+X|z|^2)}{X|z-1|^2}, \quad s_{345} = |q_{1\perp}|^2 \frac{p_3^+}{p_6^+} \frac{(|w|^2 + X|z|^2)}{|z-1|^2|w|^2}.$$

In particular, these variables clarify the physical interpretation of the singularities of the CEV:



$$A^{gg(0)}(q_1, p_4^\oplus, p_5^\oplus, q_3) = \frac{q_{1\perp}^*}{q_{1\perp}} \frac{z(w-1)(z-1)X}{(w+Xz)}$$

$$A^{gg(0)}(q_1, p_4^\oplus, p_5^\ominus, q_3) = \frac{\bar{w}^2 z^2 |z-1|^2 X^2}{|w+Xz|^2(|w|^2 + X|z|^2)} + \frac{z^2 X}{(1+X|z|^2)} + \frac{Xz}{|w+Xz|^2}\left(\frac{|w-1|^2 X \bar{z}}{(1+X)} + \bar{w}(1+X|z|^2) - (|w|^2 + X|z|^2) - \bar{w} z (1+X) \right).$$

Before we leave tree level 2: BCFW Representations

The previous representation of the vertex was derived from ref. [17a]. Since then new representations have been derived using the BCFW recursion relations, e.g. [17b].

$$M_{6g}^{(0)}\left(1^{\oplus},2^{\oplus},3^{\ominus},4^{\oplus},5^{\ominus},6^{\ominus}\right) = -\frac{\langle 3|1+2|4]^4}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]s_{123}\langle 1|2+3|4]\langle 3|4+5|6]} - \frac{\langle 5\ 6\rangle^4[2\ 4]^4}{\langle 5\ 6\rangle\langle 6\ 1\rangle[2\ 3][3\ 4]s_{234}\langle 5|6+1|2]\langle 1|2+3|4]} - \frac{\langle 3\ 5\rangle^4[1\ 2]^4}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle s_{345}\langle 3|4+5|6]\langle 5|6+1|2]}.$$

These individual terms have unphysical singularities, e.g.: $\langle 3|4+5|6] \xrightarrow[\text{NMRK}]{} -q_{3\perp}^{*}\sqrt{\frac{p_3^+}{p_6^+}}\,(p_{4\perp}+p_{5\perp})$

We can relate these spinor strings to Mandelstam invariants: $\langle 3|4+5|6]\langle 6|4+5|3] = s_{123}s_{345}\left(1-\frac{s_{12}s_{45}}{s_{123}s_{345}}\right)$

At these unphysical singularity surfaces, a corresponding cross ratio tends to unity:

$$P_u^A = p_{4\perp}+p_{5\perp} \qquad u_A = \frac{s_{12}s_{45}}{s_{123}s_{345}} \xrightarrow[\text{NMRK}]{} \frac{s_{45}}{(p_4^+ + p_5^+)(p_4^- + p_5^-)} \xrightarrow[P_u^A\to 0]{} 1$$

[17a] Nucl. Phys. B 298 (1988) Mangano, Parke, Xu; [17b] arXiv:hep-th/0412308 Britto, Cachazo, Feng

Before we leave tree level 2: BCFW Representations

In the NMRK limit, these BCFW representations lead to new representations for the opposite-helicity vertex:

$$A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) = R^A_{uv} + R^A_{vw} + R^A_{wu}\,, \qquad A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) = R^A_{\bar{u}\bar{v}} + R^A_{\bar{v}\bar{w}} + R^A_{\bar{w}\bar{u}}$$

$$R^A_{uv} = \frac{X^3|w-1|^2(\bar{z}-1)|z|^2}{(X+1)(\bar{w}+X\bar{z})(w-z)(1+X\bar{z})}\,,$$

$$R^A_{vw} = \frac{zX(w-1)(z-1)}{(1+X|z|^2)(1+X\bar{z})(w+X|z|^2)}\,,$$

$$R^A_{wu} = -\frac{z^3X^3(\bar{w}-1)|z-1|^2|z|^2}{(w+Xz)(|w|^2+X|z|^2)(w+X|z|^2)(w-z)}\,.$$

$$R^A_{\bar{u}\bar{v}} = \frac{X|w-1|^2(z-1)|z|^2}{(X+1)(w+Xz)(\bar{w}-\bar{z})(1+Xz)}\,,$$

$$R^A_{\bar{v}\bar{w}} = \frac{X^3(\bar{w}-1)(\bar{z}-1)z^4\bar{z}}{(1+X|z|^2)(1+Xz)(\bar{w}+X|z|^2)}\,,$$

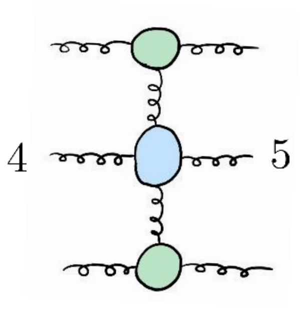
$$R^A_{\bar{w}\bar{u}} = -\frac{Xz(w-1)\bar{w}^4|z-1|^2}{(\bar{w}+X\bar{z})(|w|^2+X|z|^2)(\bar{w}+X|z|^2)(\bar{w}-\bar{z})}\,.$$

In addition to the physical singularities, we recognise the unphysical surfaces

$$\langle 1|2+3|4] = 0 \quad \leftrightarrow \quad P^A_v = 0 \quad \leftrightarrow \quad 1+X\bar{z} = 0\,,$$
$$\langle 3|4+5|6] = 0 \quad \leftrightarrow \quad P^A_u = 0 \quad \leftrightarrow \quad w-z = 0\,,$$
$$\langle 5|6+1|2] = 0 \quad \leftrightarrow \quad P^A_w = 0 \quad \leftrightarrow \quad w+X|z|^2 = 0\,,$$

Before we leave tree level 3: The B ordering

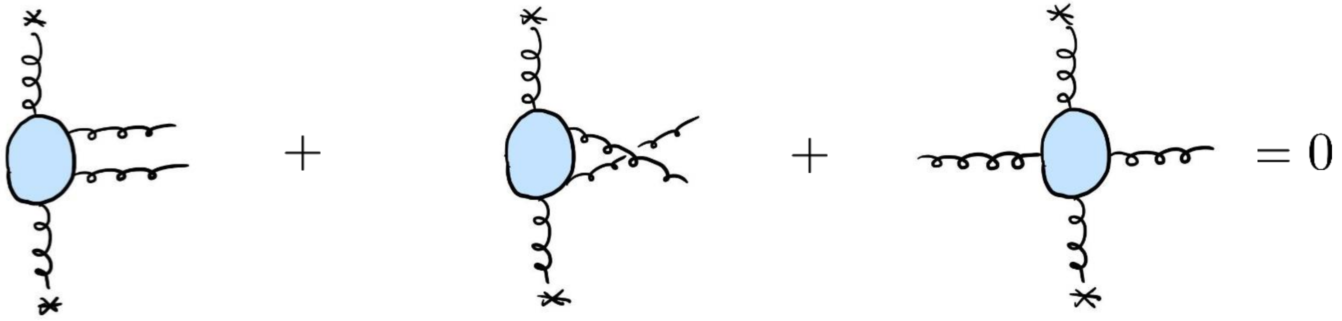
The orderings σ_A and $\sigma_{A'}$ are not the only orderings which give rise to leading behaviour in the NMRK.



$$M_{6g}^{(0)}(p_1^{\nu_1}, p_4^{\nu_4}, p_2^{\nu_2}, p_3^{\nu_3}, p_5^{\nu_5}, p_6^{\nu_6}) \xrightarrow[\text{NMRK}]{} s\, C^{g(0)}(p_2^{\nu_2}, p_3^{\nu_3}) \frac{1}{t_1} B^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) \frac{1}{t_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \,,$$

We will refer to this ordering as $\sigma_B = \{1,4,2,3,5,6\}$

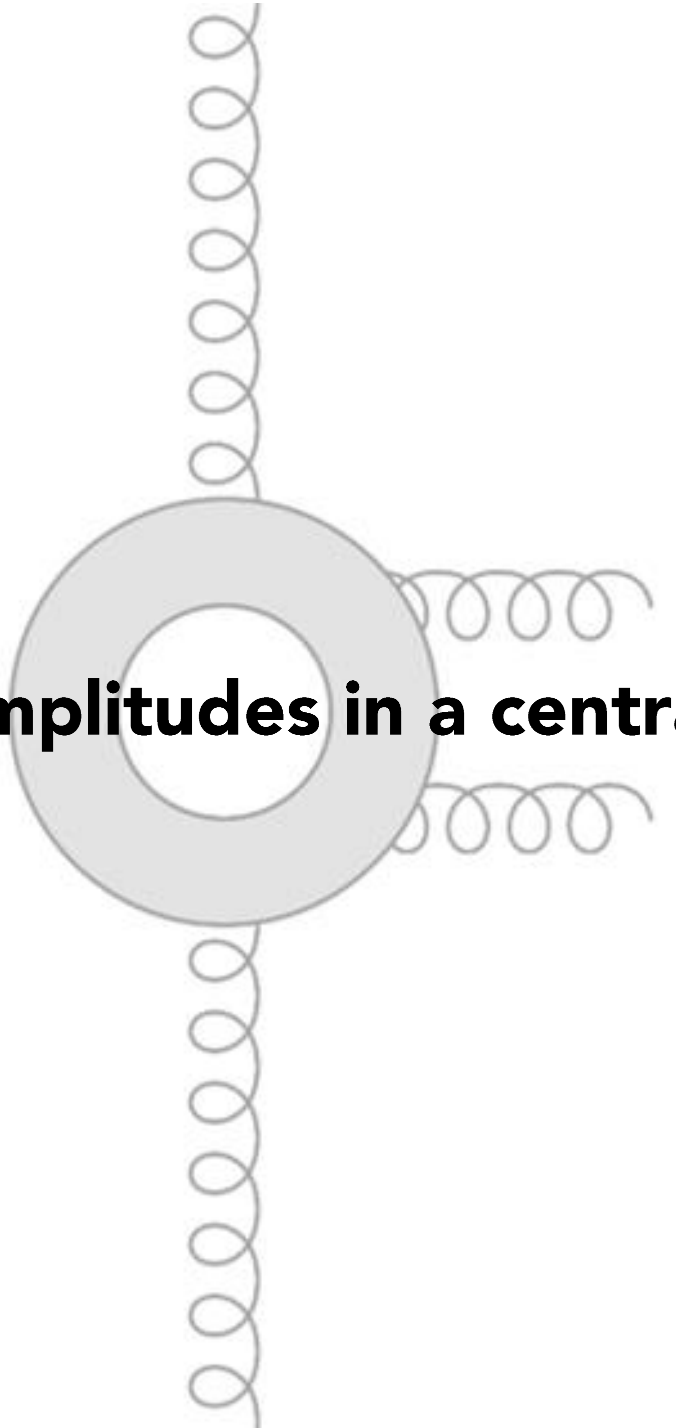
There are only 2 independent colour structures due to the relationship:



$$A^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) + A^{gg(0)}(q_1, p_5^{\nu_5}, p_4^{\nu_4}, q_3) + B^{gg(0)}(q_1, p_4^{\nu_4}, p_5^{\nu_5}, q_3) = 0.$$

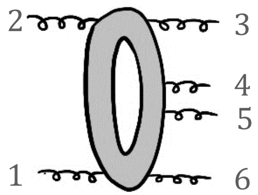
This relationship won't hold at loop level, and we will need to study all three colour structures.

3. One-loop gluon amplitudes in a central NMRK limit



Organisation of the one-loop amplitude

To obtain a central-emission vertex for two gluons we can take the central NMRK limit of the one-loop result.



We separately consider the contributions to the QCD amplitude where there is a gluon or quark circulating in the loop. We use the DDM colour basis [15]:

$$\mathcal{M}^{(1)}_{\rm QCD} = \mathcal{M}^{(1)[8]}_{\rm QCD} + \mathcal{M}^{(1)[3]}_{\rm QCD} = \sum_{\sigma\in S_5/\mathcal{R}} {\rm tr}\left(F^{a_{\sigma_1}}\cdots F^{a_{\sigma_6}}\right) M^{(1)[8]}_{\rm QCD}(\sigma_1,\ldots,\sigma_6) + \sum_{\sigma\in S_5/\mathcal{R}} 2N_f {\rm tr}\left(\lambda^{a_{\sigma_1}}\cdots\lambda^{a_{\sigma_6}}\right) M^{(1)[3]}_{\rm QCD}(\sigma_1,\ldots,\sigma_6)$$

$$F^b_{ac} = i\sqrt{2} f^{abc}$$

We further make use of the 'supersymmetric decomposition' of QCD at one loop

$$M^{(1)[8]}_{\rm QCD} = M^{(1)}_{\mathcal{N}=4} - 4M^{(1)}_{\mathcal{N}=1\chi} + M^{(1)}_{\rm scalar} \qquad M^{(1)[3]}_{\rm QCD} = M^{(1)}_{\mathcal{N}=1\chi} - M^{(1)}_{\rm scalar}$$

We extend this decomposition to colour-dressed amplitudes so that we can write the full QCD amplitude as

$$\mathcal{M}^{(1)}_{\rm QCD} = \left(\mathcal{M}^{(1)}_{\mathcal{N}=4} - 4\mathcal{M}^{(1)[8]}_{\mathcal{N}=1\chi} + \mathcal{M}^{(1)[8]}_{\rm scalar}\right) + \left(\mathcal{M}^{(1)[3]}_{\mathcal{N}=1\chi} - \mathcal{M}^{(1)[3]}_{\rm scalar}\right).$$

[15] hep-ph/9910563 Del Duca, Dixon, Maltoni

Colour exchanges in the t channel

Another way we can decompose the amplitude is through the exchange of a symmetric or antisymmetric representation in the (t_1, t_3) channel:

$$\mathcal{M}^{(1)} = \mathcal{M}^{(1)(+,+)} + \mathcal{M}^{(1)(+,-)} + \mathcal{M}^{(1)(-,+)} + \mathcal{M}^{(1)(-,-)}$$

When there is an adjoint representation circulating in the loop, we get e.g.

$$\begin{aligned}
\mathcal{M}^{(1)(-,-)} = g_s^6 F^{a_3 a_2 c_1} F^{a_6 a_1 c_3} \\
\times \frac{1}{4}\Big\{ &\mathrm{tr}\left(F^{c_1} F^{a_4} F^{a_5} F^{c_3}\right) \times \left(M^{(1)}(\sigma_A) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_A) - M^{(1)}(\underset{\leftrightarrow}{\sigma_A}) + M^{(1)}(\underset{\leftrightarrow}{\overset{\leftrightarrow}{\sigma}_A}) \right) \\
+ &\mathrm{tr}\left(F^{c_1} F^{a_5} F^{a_4} F^{c_3}\right) \times \left(M^{(1)}(\sigma_{A'}) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_{A'}) - M^{(1)}(\underset{\leftrightarrow}{\sigma_{A'}}) + M^{(1)}(\underset{\leftrightarrow}{\overset{\leftrightarrow}{\sigma}_{A'}}) \right) \\
+ &\mathrm{tr}\left(F^{c_1} F^{a_4} F^{c_3} F^{a_5}\right) \times \left(M^{(1)}(\sigma_B) - M^{(1)}(\overset{\leftrightarrow}{\sigma}_B) - M^{(1)}(\underset{\leftrightarrow}{\sigma_B}) + M^{(1)}(\underset{\leftrightarrow}{\overset{\leftrightarrow}{\sigma}_B}) \right) \Big\}
\end{aligned}$$

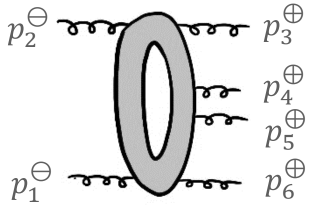
We use the shorthand notation of $\leftrightarrow$ to indicate the interchange of $p_1 \leftrightarrow p_6$ or $p_2 \leftrightarrow p_3$.

One-loop six-gluon MHV amplitude in N=4

For the remainder of this talk, our goal is to study the central NMRK limit of

$$\mathcal{M}^{(1)}_{\mathcal{N}=4}(p_1^{\nu_1},p_2^{\nu_2},p_3^{\nu_3},p_4^{\nu_4},p_5^{\nu_5},p_6^{\nu_6}) = \sum_{\sigma\in S_5/\mathcal{R}} \mathrm{tr}\,(F^{a_{\sigma_1}}\cdots F^{a_{\sigma_6}})\, M^{(1)}_{\mathcal{N}=4}\,(p_{\sigma_1}^{\nu_{\sigma_1}},\ldots,p_{\sigma_6}^{\nu_{\sigma_6}}),$$

To obtain the same-sign helicity vertex we can start from a MHV helicity configuration, e.g:



$$\mathcal{M}^{(1)}_{\mathcal{N}=4}(p_1^{\ominus},p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},p_5^{\oplus},p_6^{\oplus})$$

The N= 4 colour-ordered MHV amplitudes are $M^{(1)}_{\mathcal{N}=4}(\sigma_1,\cdots,\sigma_6) = \kappa_\Gamma M_6^{(0)}(\sigma_1,\cdots,\sigma_6)V_6(\sigma_1,\cdots,\sigma_6)$

with [18]:

$$V_6(\sigma_1,\cdots,\sigma_6) = \sum_{i=1}^{6} -\frac{1}{\epsilon}\left(\frac{\mu^2}{-t_i^{[2]}}\right)^{\epsilon} - \sum_{i=1}^{6} \ln\left(\frac{-t_i^{[2]}}{-t_i^{[3]}}\right)\ln\left(\frac{-t_{i+1}^{[2]}}{-t_i^{[3]}}\right)$$
$$+\sum_{i=1}^{3}\ln^2\left(\frac{-t_i^{[3]}}{-t_{i+1}^{[3]}}\right) - \sum_{i=1}^{3}\mathrm{Li}_2\left[1-\frac{t_i^{[2]}t_{i+3}^{[2]}}{t_i^{[3]}t_{i-1}^{[3]}}\right] + \pi^2.$$

$$t_i^{[r]} = (p_{\sigma_i}+\cdots+p_{\sigma_{i+r-1}})^2$$

Let us first apply what we know about the tree level gluon amplitude.

[18] hep-ph/9403226 Bern, Dixon, Dunbar, Kosower

Applying our tree-level knowledge

For the MHV amplitudes, we can factorise the tree-level amplitude. The (-,-) component becomes:

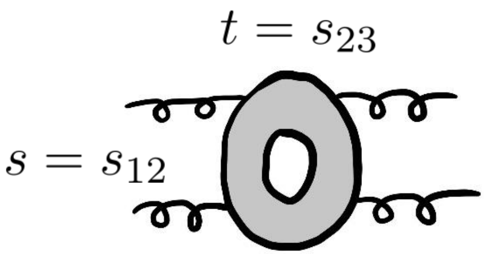
$$
\begin{aligned}
\mathcal{M}^{(1)(-,-)}_{\mathcal{N}=4}(p_4^{\nu}, p_5^{\nu}) \xrightarrow[\text{NMRK}]{} & \; g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} F^{a_3 a_2 c_1} F^{a_6 a_1 c_3} \\
& \times \frac{1}{4}\Big\{ \text{tr}\left(F^{c_1} F^{a_4} F^{a_5} F^{c_3}\right) M^{(0)}(\sigma_A)\Big|_{\text{NMRK}} \Big(V_6(\sigma_A) + V_6(\overset{\leftrightarrow}{\sigma}_A) + V_6(\underset{\leftrightarrow}{\sigma_A}) + V_6(\underset{\leftrightarrow}{\overset{\leftrightarrow}{\sigma}_A}) \Big) \\
& + \text{tr}\left(F^{c_1} F^{a_5} F^{a_4} F^{c_3}\right) M^{(0)}(\sigma_{A'})\Big|_{\text{NMRK}} \Big(V_6(\sigma_{A'}) + V_6(\overset{\leftrightarrow}{\sigma}_{A'}) + V_6(\underset{\leftrightarrow}{\sigma_{A'}}) + V_6(\underset{\leftrightarrow}{\overset{\leftrightarrow}{\sigma}_{A'}}) \Big) \\
& + \text{tr}\left(F^{c_1} F^{a_4} F^{c_3} F^{a_5}\right) M^{(0)}(\sigma_B)\Big|_{\text{NMRK}} \Big(V_6(\sigma_B) + V_6(\overset{\leftrightarrow}{\sigma}_B) + V_6(\underset{\leftrightarrow}{\sigma_B}) + V_6(\underset{\leftrightarrow}{\overset{\leftrightarrow}{\sigma}_B}) \Big) \Big\}
\end{aligned}
$$

We will see that the real parts of these transcendental functions are equal in NMRK, leading to

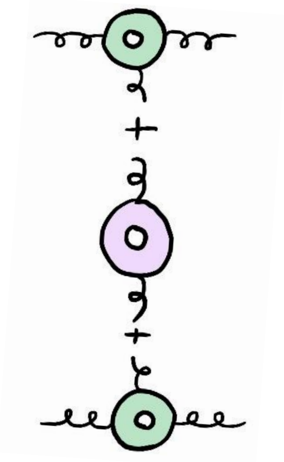
$$
\begin{aligned}
\text{Disp}\mathcal{M}^{(1)}_{\mathcal{N}=4}(p_4^{\oplus}, p_5^{\oplus}) \xrightarrow[\text{NMRK}]{} & \; g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2} s_{12} F^{a_3 a_2 c_1} C^{g(0)}\left(p_2^{\nu_2}, p_3^{-\nu_2}\right) \frac{1}{t_1}\frac{1}{t_3} F^{a_6 a_1 c_3} C^{g(0)}\left(p_1^{\nu_1}, p_6^{-\nu_1}\right) \times \\
& \Big\{ \text{tr}\left(F^{c_1} F^{a_4} F^{a_5} F^{c_3}\right) A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) \text{Re}\left[V_6(\sigma_A)|_{\text{NMRK}} \right] \\
& + \text{tr}\left(F^{c_1} F^{a_5} F^{a_4} F^{c_3}\right) A^{gg(0)}(q_1, p_5^{\oplus}, p_4^{\oplus}, q_3) \text{Re}\left[V_6(\sigma_{A'})|_{\text{NMRK}} \right] \\
& + \text{tr}\left(F^{c_1} F^{a_5} F^{c_3} F^{a_4}\right) B^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) \text{Re}\left[V_6(\sigma_B)|_{\text{NMRK}} \right] \Big\}.
\end{aligned}
$$

Lessons from four and five gluon MHV amplitudes

Before we study the six-gluon MHV transcendental function let us start with some observations.
The four-gluon function can be written *exactly* as a sum of the one-loop MRK functions [19]:

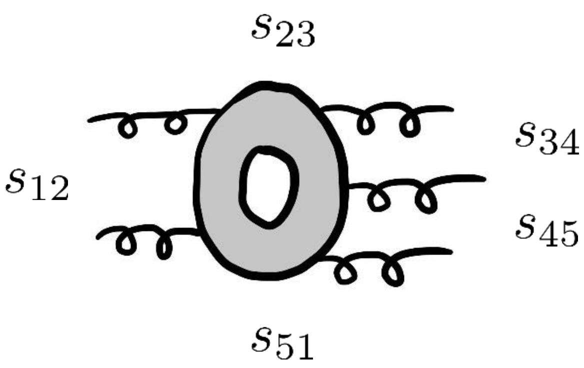


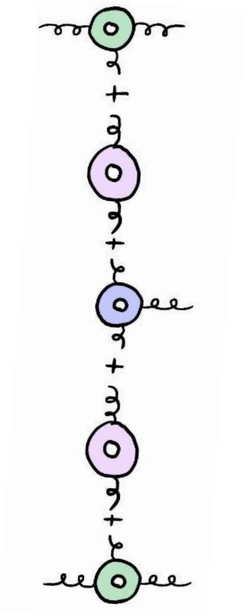
$$V_4(\sigma_1, \cdots, \sigma_5) = -\frac{2}{\epsilon}\left(\left(\frac{\mu^2}{-s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right) + \ln^2\left(\frac{-s}{-t}\right) + \pi^2$$
$$= c^{g(1)}(t) + \alpha^{(1)}(t)\log\left(\frac{s}{\tau}\right) + c^{g(1)}(t)$$



The same is true for the five-gluon MHV transcendental function [19]:

$$V_5(\sigma_1, \cdots, \sigma_5) = \sum_{i=1}^{5} -\frac{1}{\epsilon}\left(\frac{\mu^2}{-t_i^{[2]}}\right)^{\epsilon} - \sum_{i=1}^{5}\ln\left(\frac{-t_i^{[2]}}{-t_{i+1}^{[2]}}\right)\ln\left(\frac{-t_{i+1}^{[2]}}{-t_i^{[2]}}\right) + \frac{5\pi^2}{6}$$
$$= c^{g(1)}(s_{23}) + \alpha^{(1)}(s_{23})\log\left(\frac{s_{34}}{\tau}\right)$$
$$+ v^{g(1)}\left(s_{23}, \frac{s_{34}s_{45}}{s_{12}}, s_{51}\right)$$
$$+ \alpha^{(1)}(s_{51})\log\left(\frac{s_{45}}{\tau}\right) + c^{g(1)}(s_{51})$$

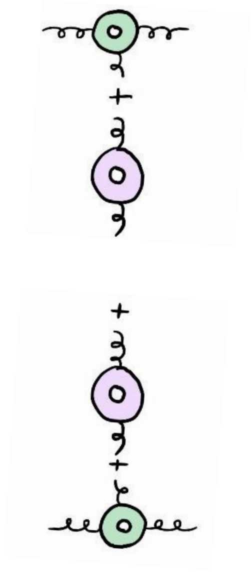




[19] hep-th/0802.2065 Lipatov, Bartels, Sabio Vera

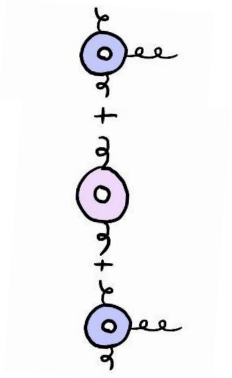
Organising the real parts of the transcendental functions

Can we do the same rewriting for the six-gluon MHV transcendental function? We first introduce a function which collects the expected 'extremal' or 'non-central' functions:



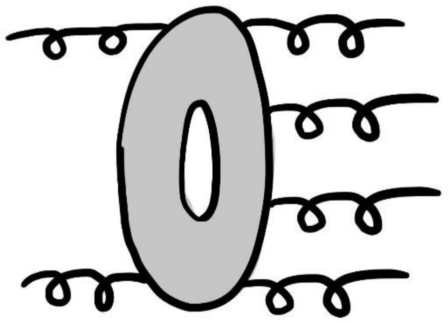
$$E(\tilde{t}_1, \tilde{s}_1; \tilde{t}_3, \tilde{s}_3) = c^{g(1)}_{\mathcal{N}=4}(\tilde{t}_1; \tau) + \alpha^{(1)}(\tilde{t}_1) \log\left(\frac{\tilde{s}_1}{\tau}\right) + \alpha^{(1)}(\tilde{t}_3) \log\left(\frac{\tilde{s}_3}{\tau}\right) + c^{g(1)}_{\mathcal{N}=4}(\tilde{t}_3; \tau)\,.$$

and another function which collects the central MRK functions:



$$U(\tilde{t}_1, \tilde{\eta}_{12}, \tilde{t}_2, \tilde{s}_2, \tilde{\eta}_{23}, \tilde{t}_3) = v^{g(1)}_{\mathcal{N}=4}\left(\tilde{t}_1, \tilde{\eta}_{12}, \tilde{t}_2\right) + \alpha^{(1)}(\tilde{t}_2) \log\left(\frac{\tilde{s}_2}{\tau}\right) + v^{g(1)}_{\mathcal{N}=4}\left(\tilde{t}_2, \tilde{\eta}_{23}, \tilde{t}_3\right)\,.$$

Organising the real parts of $\mathbf{V_6}(\sigma_\mathbf{A})$



Let's assign the 'non-central' variables first:

$$N_c \kappa_\Gamma \mathrm{Re} V_6(\sigma_A) = E(s_{23}, s_{34}; s_{61}, s_{56}) + V_A \,.$$

These are the unique cyclically ordered Mandelstam variables which give the correct MRK limit:

$$E(s_{23}, s_{34}; s_{61}, s_{56}) \xrightarrow[\text{(N)MRK}]{} E(-|q_{1\perp}|^2, p_3^+ p_4^-; -|q_{3\perp}|^2, p_5^+ p_6^-) \,.$$

The remaining 'central' piece may be further decomposed into the central Regge functions plus a NMRK remainder:

$$V_A = U_A + \Delta V_A(u_A, v_A, w_A) \,,$$

The variable assignment:

$$U_A = U\left(s_{23}, \frac{s_{34} s_{45}}{s_{345}\sqrt{u_A}}, s_{234}, s_{45}, \frac{s_{45} s_{56}}{s_{456}\sqrt{u_A}}, s_{61} \right) \,.$$

Lead to the *NMRK remainder function*:

$$\Delta V_A(u_A, v_A, w_A) = N_c \kappa_\Gamma \left(\frac{\pi^2}{3} - \frac{1}{4}\log^2(u_A) - \frac{1}{2}\log(u_A)\log(v_A w_A) - \mathrm{Li}\,(1-u_A) - \mathrm{Li}\,(1-v_A) - \mathrm{Li}\,(1-w_A) \right) .$$

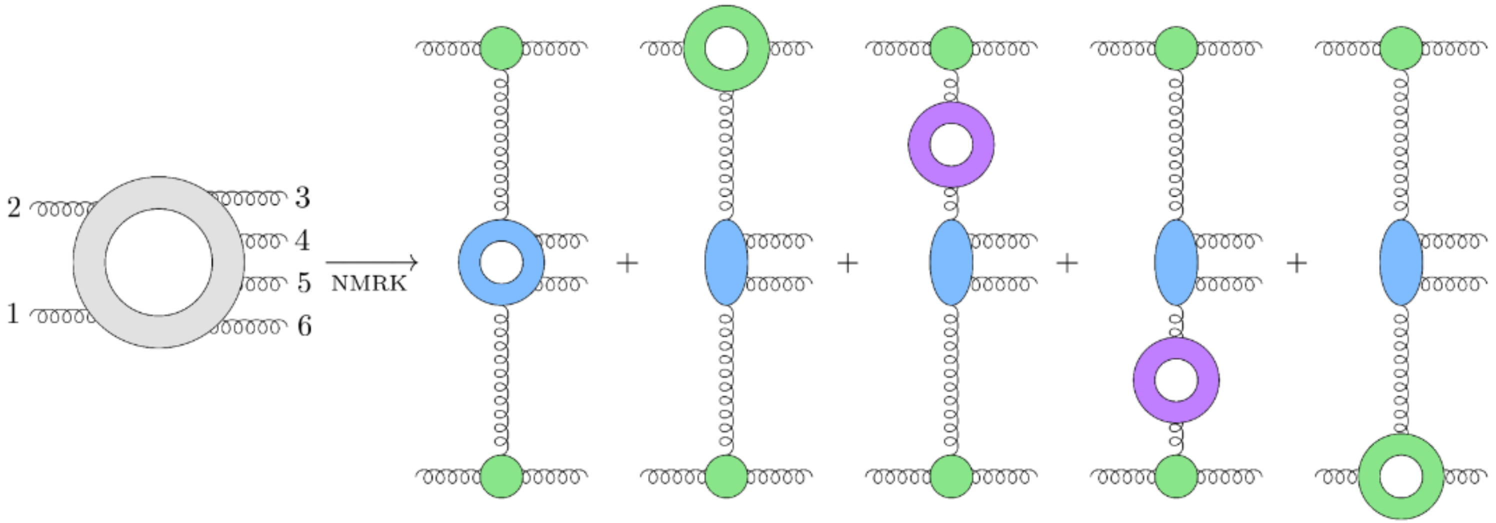
which is IR finite and vanishes in the MRK limit.

Organising the real parts of $\mathbf{V_6}(\sigma_\mathbf{A})$

With the real part of the function $V_6(\sigma_A)$ expressed in this form, we can immediately see that the in the NMRK limit,

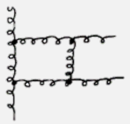
$$U_A \xrightarrow[\text{NMRK}]{} U\left(-|q_{1\perp}|^2, \frac{p_4^- \; s_{45}}{p_4^- + p_5^-}, -|q_{2\perp}|^2 - p_5^+ p_4^-, s_{45}, \frac{s_{45} \; p_5^+}{p_4^+ + p_5^+}, -|q_{3\perp}|^2\right) \; ,$$

depends only on the central degrees of freedom. This means we have achieved our desired factorisation for this colour ordering.

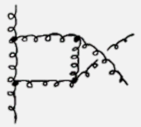


We also find that in the NMRK, $\quad \mathrm{Re}\Big[V_6(\sigma_A)\Big] = \mathrm{Re}\Big[V_6(\overset{\leftrightarrow}{\sigma}_A)\Big] = \mathrm{Re}\Big[V_6(\underset{\leftrightarrow}{\sigma_A})\Big] = \mathrm{Re}\Big[V_6(\underset{\leftrightarrow}{\overset{\leftrightarrow}{\sigma}_A})\Big]$

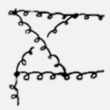
Changing basis of colour structures



T_A



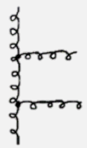
$T_{A'}$



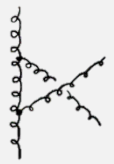
T_B

In order to connect to the tree level CEV, we move to a basis that includes the tree-level structures:

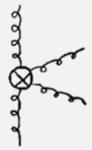
$$
\begin{aligned}
T_A M_A + T_{A'} M_{A'} + T_B M_B = {} & \frac{1}{3}(T_A - T_B)(2M_A - M_{A'} - M_B) \\
& + \frac{1}{3}(T_{A'} - T_B)(2M_{A'} - M_A - M_B) \\
& + \frac{1}{3}(T_A + T_{A'} + T_B)(M_A + M_{A'} + M_B)
\end{aligned}
$$

N_c 

$(T_A - T_B)$

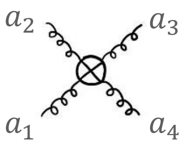
N_c 

$(T_{A'} - T_B)$

3 

$(T_A + T_{A'} + T_B)$

We find that in addition to the tree-level structures, we have an additional totally symmetric colour structure:



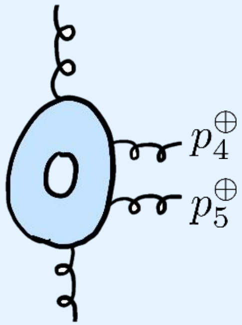
$$= d_A^{a_1 a_2 a_3 a_4} = \frac{1}{4!} \sum_{S_4} \mathrm{tr}\left(F^{a_{\sigma_1}} F^{a_{\sigma_2}} F^{a_{\sigma_3}} F^{a_{\sigma_4}}\right)$$

Extracting the same-helicity CEV

In this new colour basis, we write the dispersive part of the amplitude as

$$\text{Disp}\left[\mathcal{M}^{(1)}_{\mathcal{N}=4}(p_4^{\oplus}, p_5^{\oplus})\right] \xrightarrow[\text{NMRK}]{} \frac{g_s^6}{(4\pi)^2}\, s_{12} F^{a_3 a_2 c_1} C^{g(0)}\left(p_2^{\nu_2}, p_3^{-\nu_2}\right) \frac{1}{t_1} \times$$
$$\sum_{\sigma\in S_2}\left\{(F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3}\left(A_{\vDash}^{gg(1)}(q_1, p_{\sigma_4}^{\oplus}, p_{\sigma_5}^{\oplus}, q_3) + A^{gg(0)}(q_1, p_{\sigma_4}^{\oplus}, p_{\sigma_5}^{\oplus}, q_3) E(t_1, s_{3\sigma_4}; t_3, s_{\sigma_5 6})\right) + \frac{1}{N_c} d_A^{c_1 a_{\sigma_4} a_{\sigma_5} c_3} A_d^{gg(1)}(q_1, p_{\sigma_4}^{\oplus}, p_{\sigma_5}^{\oplus}, q_3)\right\}$$
$$\times \frac{1}{t_3} F^{a_6 a_1 c_3} C^{g(0)}\left(p_1^{\nu_1}, p_6^{-\nu_1}\right) ,$$

The one loop coefficients of these colour structures are



$$A_{\vDash}^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) = \frac{1}{3}\left(A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3)\,(2V_A + V_B) - A^{gg(0)}(q_1, p_5^{\oplus}, p_4^{\oplus}, q_3)\,(V_{A'} - V_{B'})\right) .$$

$$A_d^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) = A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3)\,(V_A - V_B) .$$

Taking the further MRK limit

The notation introduced here makes it simple to take the MRK limit of the amplitude.

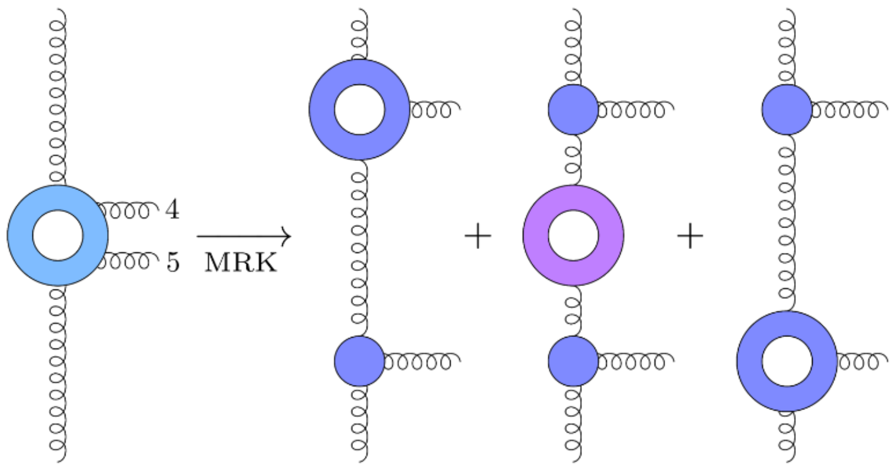
The one-loop MRK functions tend to their expected MRK limit by construction:

$$U_A \xrightarrow[\text{MRK}]{} U_{\text{MRK}} = U(-|q_{1\perp}|^2, |p_{4\perp}|^2, -|q_{2\perp}|^2, p_4^+ p_5^-, |p_{5\perp}|^2, -|q_{3\perp}|^2).$$

while the NMRK remainder functions vanish. This leads to

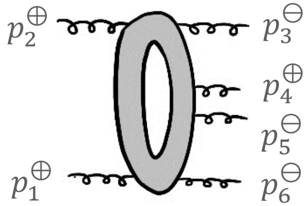
$$A_{\vDash}^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) \xrightarrow[\text{MRK}]{} V^{g(0)}(q_1, p_4^{\oplus}, q_2)\frac{1}{t_2}V^{g(0)}(q_2, p_5^{\oplus}, q_3)\, U_{\text{MRK}}\,,$$

$$A_{d}^{gg(1)}(q_1, p_4^{\oplus}, p_5^{\oplus}, q_3) \xrightarrow[\text{MRK}]{} 0\,.$$



Opposite helicity amplitude

To obtain the opposite-sign helicity vertex we must start from a NMHV helicity configuration, e.g:



$$\mathcal{M}^{(1)}_{\mathcal{N}=4}(p_1^{\oplus}, p_2^{\oplus}, p_3^{\ominus}, p_4^{\oplus}, p_5^{\ominus}, p_6^{\ominus})$$

The NMHV colour-ordered amplitudes are not directly proportional to the tree-level amplitude [20]:

$$\begin{aligned}\frac{(4\pi)^2}{\kappa_\Gamma} M^{(1)}_{\mathcal{N}=4}\left(1^{\ominus}, 2^{\ominus}, 3^{\oplus}, 4^{\ominus}, 5^{\oplus}, 6^{\oplus}\right) &= \left(\frac{\langle 12\rangle^4 [56]^4}{D_1} + \frac{\langle 4|1+2|3]^4}{D_1^*}\right) W_1(\sigma) \\ &+ \left(\frac{\langle 1|2+4|3]^4}{D_2^*} + \frac{\langle 24\rangle^4 [56]^4}{D_2}\right) W_2(\sigma) \\ &+ \left(\frac{\langle 4|1+2|6]^4}{D_3} + \frac{\langle 12\rangle^4 [35]^4}{D_3^*}\right) W_3(\sigma)\,,\end{aligned}$$

where the denominators are cyclic permutations of $D_1 = -\langle 1\ 2\rangle\langle 2\ 3\rangle\,[4\ 5]\,[5\ 6]\,s_{123}\langle 1|2+3|4]\langle 3|4+5|6]\,.$

The rational coefficients in the NMRK will be given by the pairs

$$S^I_{xy} = R^I_{xy} + R^I_{\bar{x}\bar{y}} \qquad x, y \in \{u, v, w\},\ I \in \{A, A', B\}.$$

[20] hep-ph/9409265 Bern, Dixon, Dunbar, Kosower

One-loop six-gluon NMHV amplitude in N=4

The NMHV transcendental function is [20]:

$$W_i(\sigma) = -\frac{1}{2\epsilon^2}\sum_{j=1}^{6}\left(\frac{\mu^2}{-t_j^{[2]}}\right)^{\epsilon} - \log\left(\frac{-t_i^{[3]}}{-t_i^{[2]}}\right)\log\left(\frac{-t_i^{[3]}}{-t_{i+1}^{[2]}}\right) - \log\left(\frac{-t_i^{[3]}}{-t_{i+3}^{[2]}}\right)\log\left(\frac{-t_i^{[3]}}{-t_{i+4}^{[2]}}\right)$$
$$+ \log\left(\frac{-t_i^{[3]}}{-t_{i+2}^{[2]}}\right)\log\left(\frac{-t_i^{[3]}}{-t_{i+5}^{[2]}}\right) + \frac{1}{2}\log\left(\frac{-t_i^{[2]}}{-t_{i+3}^{[2]}}\right)\log\left(\frac{-t_{i+1}^{[2]}}{-t_{i+4}^{[2]}}\right)$$
$$+ \frac{1}{2}\log\left(\frac{-t_{i+5}^{[2]}}{-t_i^{[2]}}\right)\log\left(\frac{-t_{i+1}^{[2]}}{-t_{i+2}^{[2]}}\right) + \frac{1}{2}\log\left(\frac{-t_{i+2}^{[2]}}{-t_{i+3}^{[2]}}\right)\log\left(\frac{-t_{i+4}^{[2]}}{-t_{i+5}^{[2]}}\right) + \frac{\pi^2}{3}.$$

where we use the shorthand $W_i(\sigma) = W_i(\sigma_i, \sigma_{i+1}, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}, \sigma_{i+5})$

We note that there are no dilogarithms in this function.

[20] hep-ph/9409265 Bern, Dixon, Dunbar, Kosower

Reorganising the NMHV amplitude

As before, we limit ourselves to the dispersive part of the amplitude, where only the (-,-) component contributes:

$$
\begin{aligned}
&\mathrm{Disp}\left[\mathcal{M}^{(1)}_{\mathcal{N}=4}(p_4^{\oplus}, p_5^{\ominus})\right] \xrightarrow[\text{NMRK}]{} \\
&g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2}\, s_{12}\, F^{a_3 a_2 c_1}\, C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2})\, \frac{1}{t_1}\, \frac{1}{t_3}\, F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1}) \\
&\times \Big\{ \mathrm{tr}\,(F^{c_1} F^{a_4} F^{a_5} F^{c_3}) \left[S^{A}_{uv} \mathrm{Re}\,[W_1(\sigma_A)] + S^{A}_{vw} \mathrm{Re}\,[W_2(\sigma_A)] + S^{A}_{wu} \mathrm{Re}\,[W_3(\sigma_A)]\right] \\
&\quad + \mathrm{tr}\,(F^{c_1} F^{a_5} F^{a_4} F^{c_3}) \left[S^{A'}_{uv} \mathrm{Re}\,[W_1(\sigma_{A'})] + S^{A'}_{vw} \mathrm{Re}\,[W_2(\sigma_{A'})] + S^{A'}_{wu} \mathrm{Re}\,[W_3(\sigma_{A'})]\right] \\
&\quad + \mathrm{tr}\,(F^{c_1} F^{a_5} F^{c_3} F^{a_4}) \left[S^{B}_{uv} \mathrm{Re}\,[W_1(\sigma_B)] + S^{B}_{vw} \mathrm{Re}\,[W_2(\sigma_B)] + S^{B}_{wu} \mathrm{Re}\,[W_3(\sigma_B)]\right] \Big\},
\end{aligned}
$$

Reorganising the NMHV amplitude

As before, we limit ourselves to the dispersive part of the amplitude, where only the (-,-) component contributes:

$$
\begin{aligned}
&\mathrm{Disp}\left[\mathcal{M}^{(1)}_{\mathcal{N}=4}(p_4^{\oplus}, p_5^{\ominus})\right] \xrightarrow[\mathrm{NMRK}]{} g_s^6 \frac{\kappa_\Gamma}{(4\pi)^2}\, s_{12}\, F^{a_3 a_2 c_1}\, C^{g(0)}(p_2^{\nu_2}, p_3^{-\nu_2})\, \frac{1}{t_1} \\
&\times \Big\{ \mathrm{tr}\,(F^{c_1} F^{a_4} F^{a_5} F^{c_3}) \Big[2A^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \mathrm{Re}\,[W_2(\sigma_A)] + S^A_{uv} \mathrm{Re}\,[W_1(\sigma_A) - W_2(\sigma_A)] + S^A_{wu} \mathrm{Re}\,[W_3(\sigma_A) - W_2(\sigma_A)] \Big] \\
&+ \mathrm{tr}\,(F^{c_1} F^{a_5} F^{a_4} F^{c_3}) \Big[2A^{gg(0)}(q_1, p_5^{\ominus}, p_4^{\oplus}, q_3) \mathrm{Re}\,[W_2(\sigma_{A'})] + S^{A'}_{uv} \mathrm{Re}\,[W_1(\sigma_{A'}) - W_2(\sigma_{A'})] + S^{A'}_{wu} \mathrm{Re}\,[W_3(\sigma_{A'}) - W_2(\sigma_{A'})] \Big] \\
&+ \mathrm{tr}\,(F^{c_1} F^{a_5} F^{c_3} F^{a_4}) \Big[2B^{gg(0)}(q_1, p_4^{\oplus}, p_5^{\ominus}, q_3) \mathrm{Re}\,[W_1(\sigma_B)] + S^B_{vw} \mathrm{Re}\,[W_2(\sigma_A) - W_1(\sigma_A)] + S^B_{wu} \mathrm{Re}\,[W_3(\sigma_A) - W_1(\sigma_A)] \Big] \Big\} \\
&\times \frac{1}{t_3}\, F^{a_6 a_1 c_3} C^{g(0)}(p_1^{\nu_1}, p_6^{-\nu_1})\,.
\end{aligned}
$$

Reorganising the NMHV amplitude

As before, we limit ourselves to the dispersive part of the amplitude, where only the (-,-) component contributes:

$$\mathrm{tr}\,(F^{c_1}F^{a_4}F^{a_5}F^{c_3})\left[2A^{gg(0)}(q_1,p_4^{\oplus},p_5^{\ominus},q_3)\mathrm{Re}\,[W_2(\sigma_A)]+S^A_{uv}\mathrm{Re}\,[W_1(\sigma_A)-W_2(\sigma_A)]+S^A_{wu}\mathrm{Re}\,[W_3(\sigma_A)-W_2(\sigma_A)]\right]$$

The difference between the cyclic permutations are much simpler than the full function:

$$W_1(\sigma_A)-W_2(\sigma_A)=\log(v_A)\log\left(\frac{w_A}{u_A}\right) \qquad W_3(\sigma_A)-W_2(\sigma_A)=\log(w_A)\log\left(\frac{v_A}{u_A}\right)$$

It now becomes easy to see how the amplitude is free from unphysical poles:

	$P^A_u=0$	$P^A_v=0$	$P^A_w=0$
u_A	1	$\frac{\lvert w-1\rvert^2X}{(1+X)(1+\lvert w\rvert^2X)}$	$\frac{\lvert z-1\rvert^2X}{(1+X)(1+\lvert z\rvert^2X)}$
v_A	$\frac{\lvert z-1\rvert^2X}{(1+X)(1+\lvert z\rvert^2X)}$	1	$\frac{\lvert z-1\rvert^2X}{(1+X)(1+\lvert z\rvert^2X)}$
w_A	$\frac{\lvert z-1\rvert^2X}{(1+X)(1+\lvert z\rvert^2X)}$	$\frac{\lvert w-1\rvert^2X}{(1+X)(1+\lvert w\rvert^2X)}$	1

$$S^A_{uv}\left(W_1(\sigma_A)-W_2(\sigma_A)\right)+S^A_{wu}\left(W_3(\sigma_A)-W_2(\sigma_A)\right)\xrightarrow[P^A_u\to 0]{}\left(S^A_{uv}+S^A_{wu}\right)\log^2\left(\frac{|z-1|^2X}{(1+X)(1+|z|^2X)}\right),$$

In this sum of rational terms, P^A_u is a removable singularity.

Reorganising the NMHV amplitude

The last piece we need to investigate in NMRK is the NMHV transcendental function itself.

It can be organised in the same way as the MHV amplitude, i.e. it can be written *exactly* as

$$2N_c\kappa_\Gamma \mathrm{Re}\left[W_2(\sigma_A)\right] = E(t_1, s_{34}; t_3, s_{56}) + 2W_2^A \,,$$

With the central function $$2W_2^A = U_A + \Delta W_A(u_A, v_A, w_A)$$

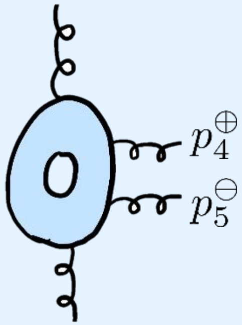
where U_A is the same *helicity independent* function that appeared in the MHV amplitude, but with NMRK remainder

$$\Delta W_A(u_A, v_A, w_A) = N_c\kappa_\Gamma\left(-\frac{1}{4}\log^2(u_A) + \frac{1}{2}\log(u_A)\log(v_A w_A) - \log(v_A)\log(w_A)\right)$$

Note that this function doesn't vanish in the MRK limit.

Extracting the opposite-helicity vertex

Performing the same change of basis of colour structures as before, we can finally extract the kinematic coefficients for the opposite-helicity vertex:

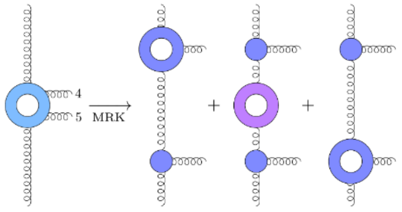


$$A_{\vDash}^{gg(1)}(q_1,p_4^{\oplus},p_5^{\ominus},q_3) = \frac{1}{3}\Big[2A^{gg(0)}(q_1,p_4^{\oplus},p_5^{\ominus},q_3)\Big(2W_2^A+W_1^B\Big) - 2A^{gg(0)}(q_1,p_5^{\ominus},p_4^{\oplus},q_3)\Big(W_2^{A'}-W_1^{B'}\Big)$$
$$+2\Big(S_{uv}^A(W_1^A-W_2^A)+S_{wu}^A(W_3^A-W_2^A)\Big) - \Big(S_{uv}^{A'}(W_1^{A'}-W_2^{A'})+S_{wu}^{A'}(W_3^{A'}-W_2^{A'})\Big)$$
$$-\Big(S_{vw}^B(W_2^B-W_1^B)+S_{wu}^B(W_3^B-W_1^B)\Big)\Big]$$

$$A_d^{gg(1)}(q_1,p_4^{\oplus},p_5^{\ominus},q_3) = 2A^{gg(0)}(q_1,p_4^{\oplus},p_5^{\ominus},q_3)\left(W_2^A-W_1^B\right)$$
$$+S_{uv}^A(W_1^A-W_2^A)+S_{wu}^A(W_3^A-W_2^A)+S_{vw}^B(W_2^B-W_1^B)$$

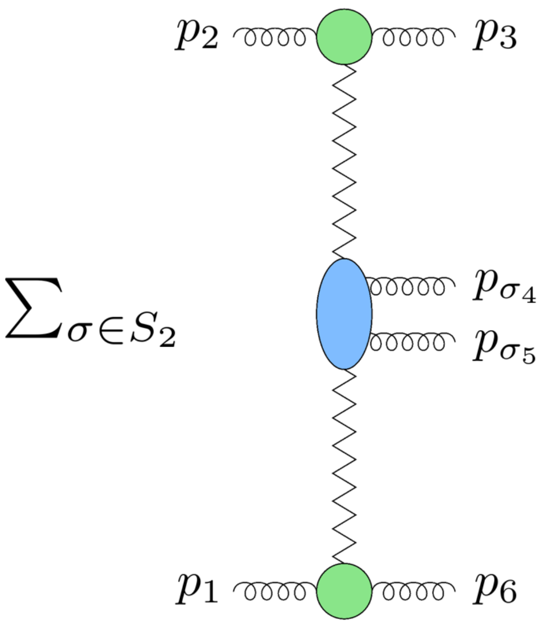
As for the same-helicity vertex, the opposite-helicity vertex tends to the expected MRK limit:

$$A_{\vDash}^{gg(1)}(q_1,p_4^{\oplus},p_5^{\ominus},q_3) \xrightarrow[\text{MRK}]{} V^{g(0)}(q_1,p_4^{\oplus},q_2)\frac{1}{t_2}V^{g(0)}(q_2,p_5^{\ominus},q_3)\,U_{\text{MRK}}\,,$$
$$A_d^{gg(1)}(q_1,p_4^{\oplus},p_5^{\ominus},q_3) \xrightarrow[\text{MRK}]{} 0\,.$$



All-orders conjecture for amplitudes in a central NMRK

The following conjecture is compatible with the one-loop gluon amplitudes, as well as the known MRK limit



$$\begin{aligned}
\mathrm{Disp}\left[\mathcal{M}_{6g}^{(-,-)}\right] \xrightarrow[\text{NMRK}]{} & \; s\left[g_s (F^{a_3})_{a_2 c_1}\, C^g(p_2^{\nu_2}, p_3^{\nu_3})\right] \\
& \times \sum_{\sigma\in S_2}\left\{\frac{1}{t_1}\frac{1}{2}\left[\left(\frac{s_{3\sigma_4}}{\tau}\right)^{\alpha(t_1)} + \left(\frac{-s_{3\sigma_4}}{\tau}\right)^{\alpha(t_1)}\right]\right. \\
& \times \left[g_s^2 (F^{a_{\sigma_4}} F^{a_{\sigma_5}})_{c_1 c_3}\, A_{\vDash}^{gg}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3) + \frac{1}{N_c} d_A^{c_1 a_{\sigma_4} a_{\sigma_5} c_3}\, A_d^{gg}(q_1, p_{\sigma_4}^{\nu_{\sigma_4}}, p_{\sigma_5}^{\nu_{\sigma_5}}, q_3)\right] \\
& \left.\times \frac{1}{t_3}\frac{1}{2}\left[\left(\frac{s_{\sigma_5 6}}{\tau}\right)^{\alpha(t_3)} + \left(\frac{-s_{\sigma_5 6}}{\tau}\right)^{\alpha(t_3)}\right]\right\} \\
& \times \left[g_s (F^{a_6})_{a_1 c_3}\, C^g(p_1^{\nu_1}, p_6^{\nu_6})\right].
\end{aligned}$$

The following conjecture is compatible with the one-loop gluon amplitudes, as well as the known MRK limit.

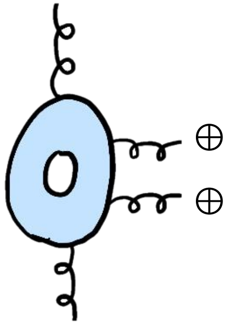
Note that this is not an exact factorisation, but rather a sum over two colour structures.

We have assumed that the new totally symmetric colour structure receives large logarithmic corrections at higher orders: strong hints will come from the 2-loop 5-gluon amplitude.

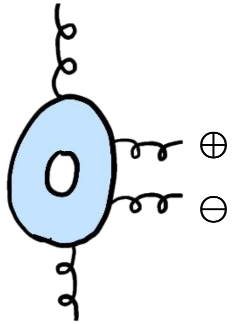
Summary

We have obtained a central-emission vertex for the emission of two gluons in a NMRK limit in N=4 SYM.

This vertex has some novel features compared to the expressions that were found at NLL:



- The presence of dilogarithms
- New one-loop colour structure
- The dressing of spurious rational terms with transcendental functions



Work is in progress to obtain the vertex in QCD.

It will be interesting to see if the Regge-limit based organisation proves useful for other amplitudes.

Thanks for your attention!