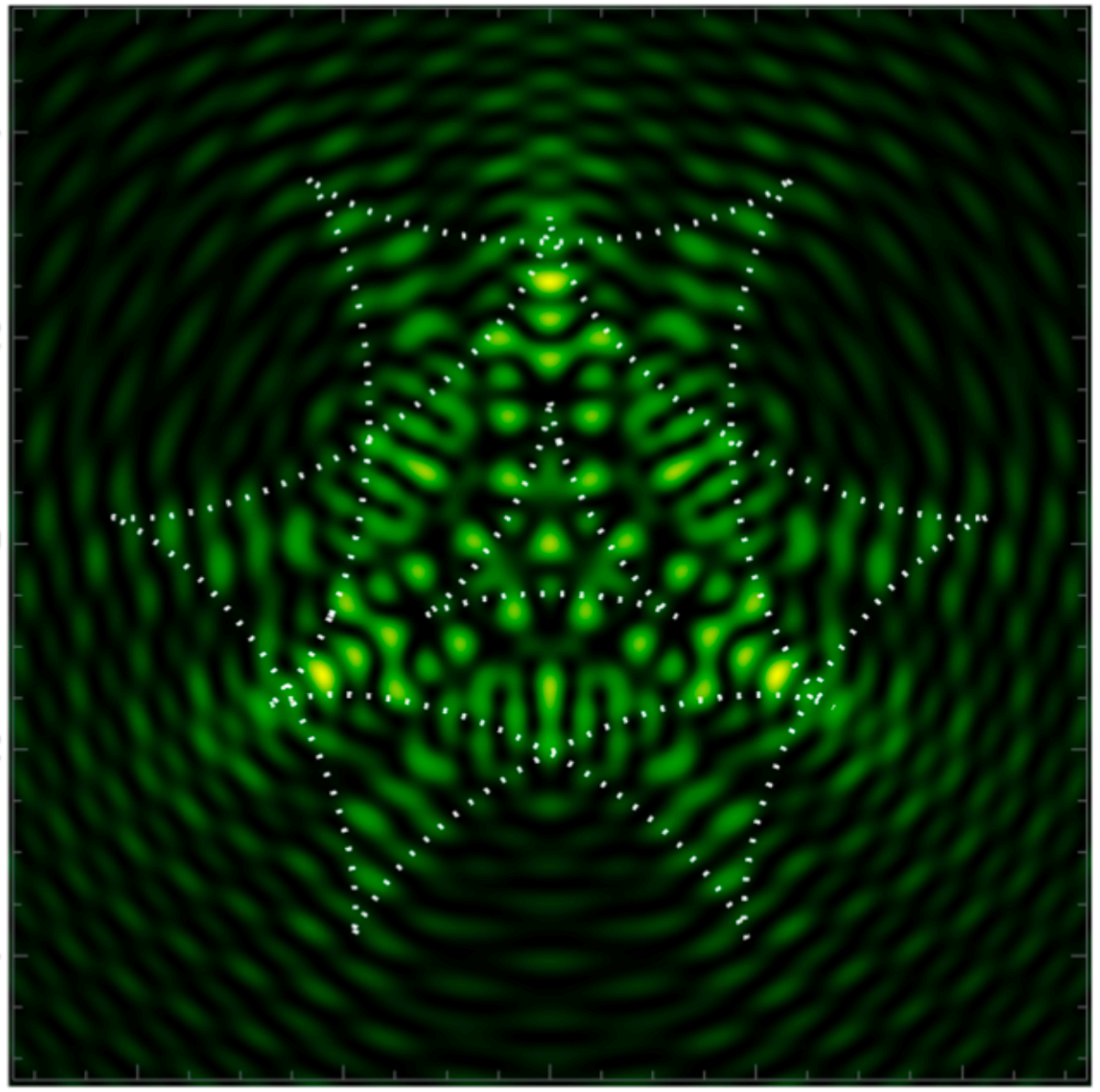
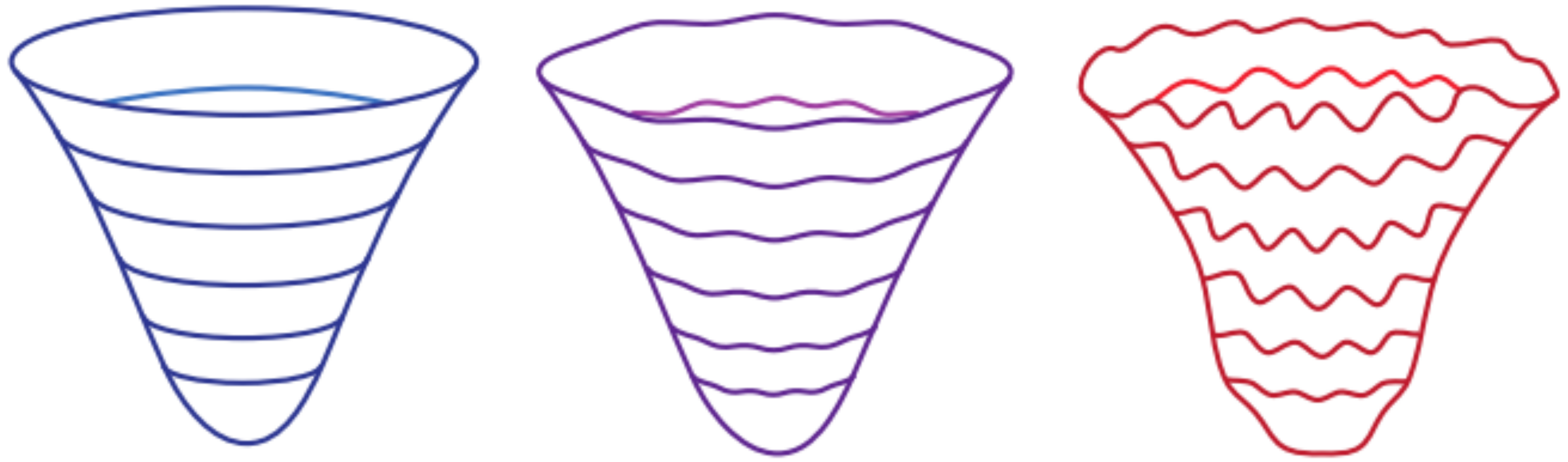
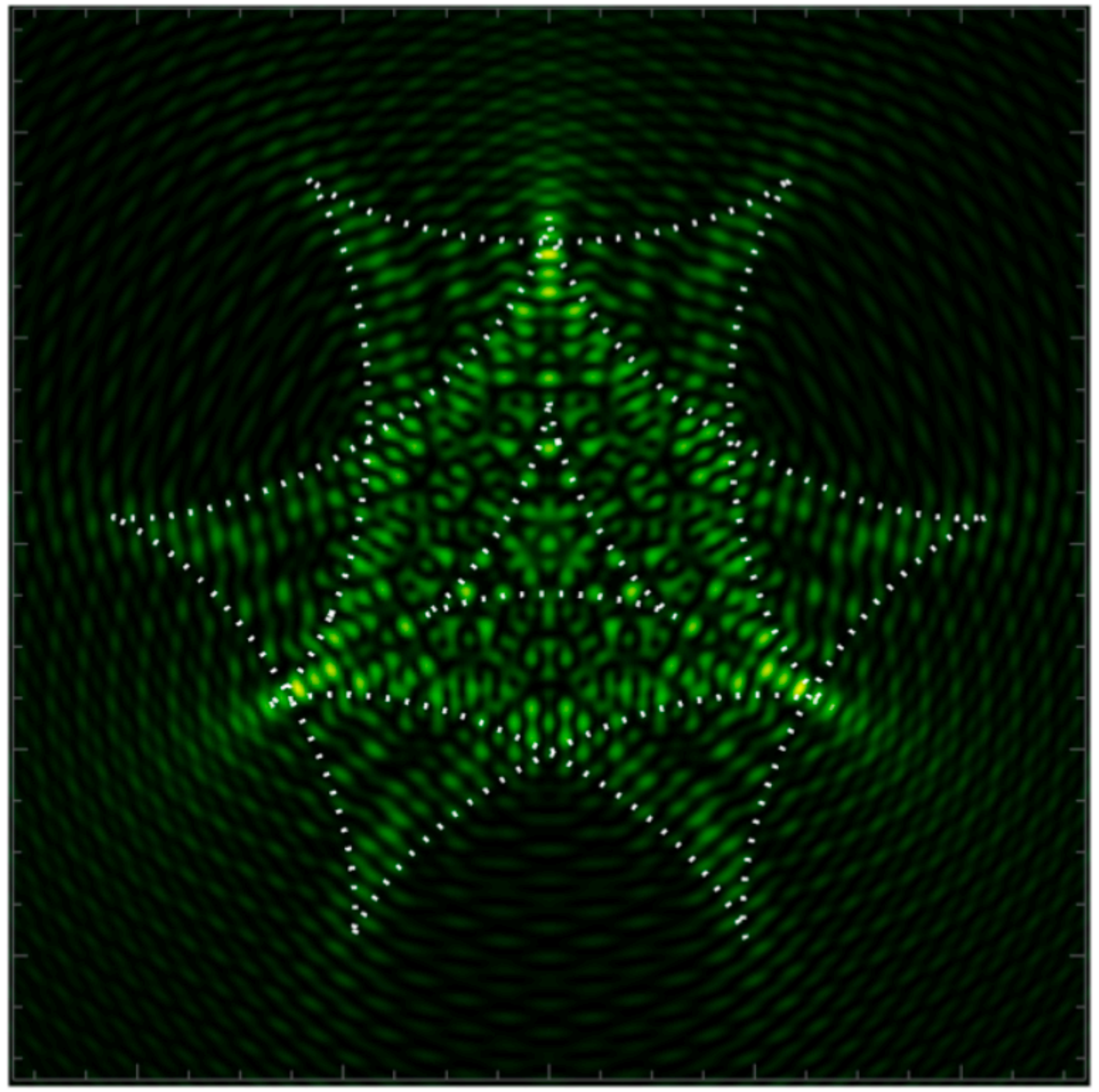
Interference, Caustics and Oscillatory Integrals







Job Feldbrugge

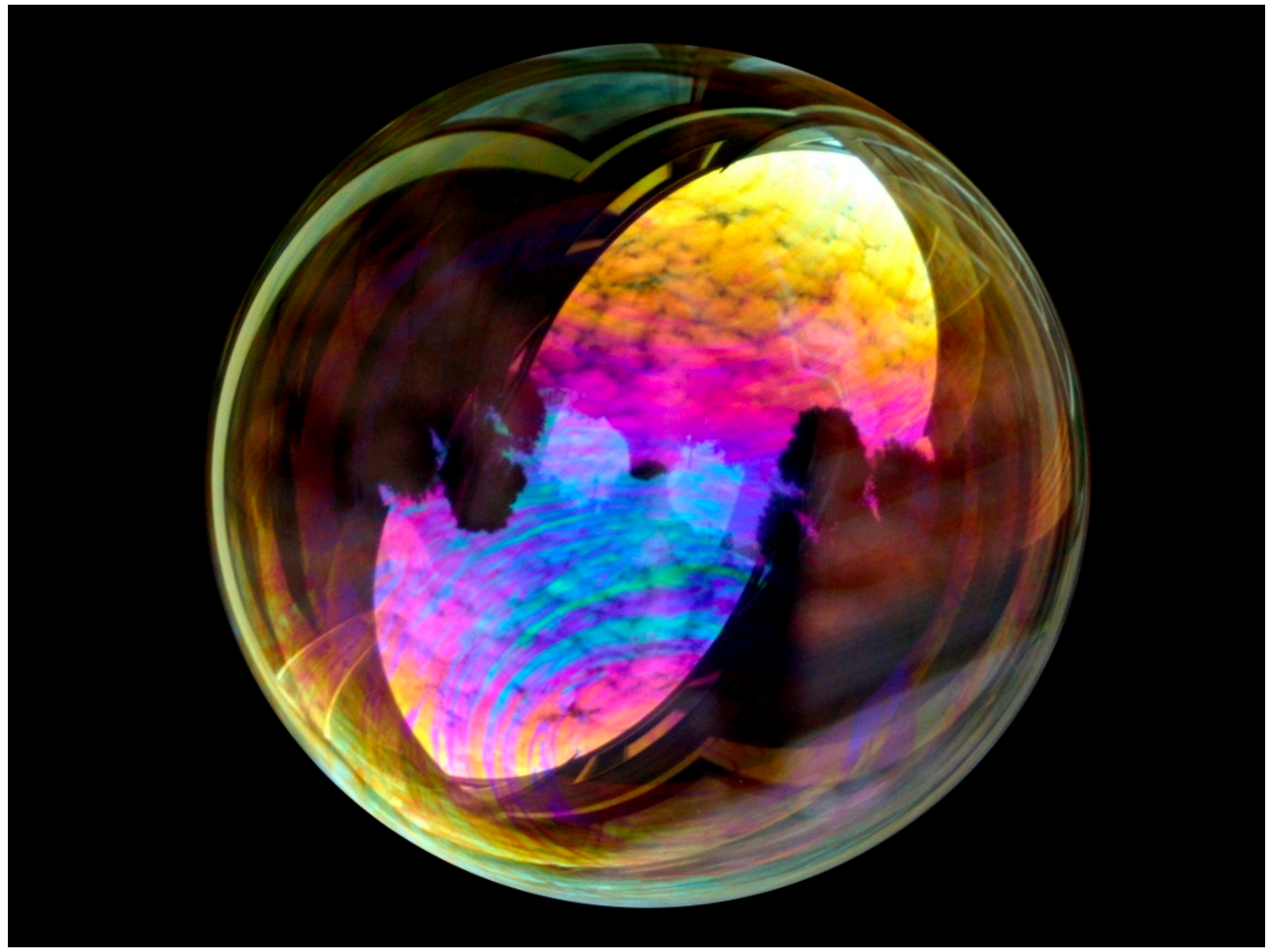
Higgs Hour: 22 October 2021

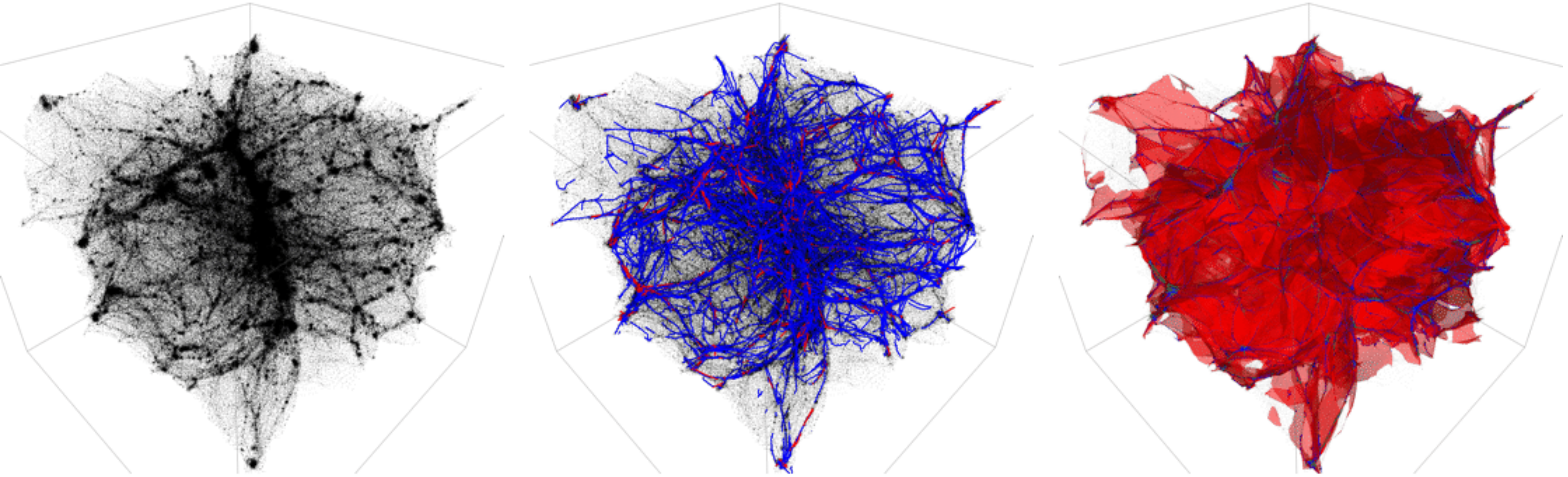
Collaborators: N Turok, U-L Pen, J-L Lehners

Caustics and Interference

Caustics and interference are among the most universal phenomena in physics

- Oscillatory integrals and quantum physics
- Caustics in large scale structure formation





I am going to give […] an ***elementary demonstration****. But elementary does* ***not mean easy to understand****. Elementary means that very little is required to know ahead of time in order to understand it, except to have* ***an infinite amount of intelligence****. - Richard P. Feynman (1964)*

Oscillatory integrals

Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

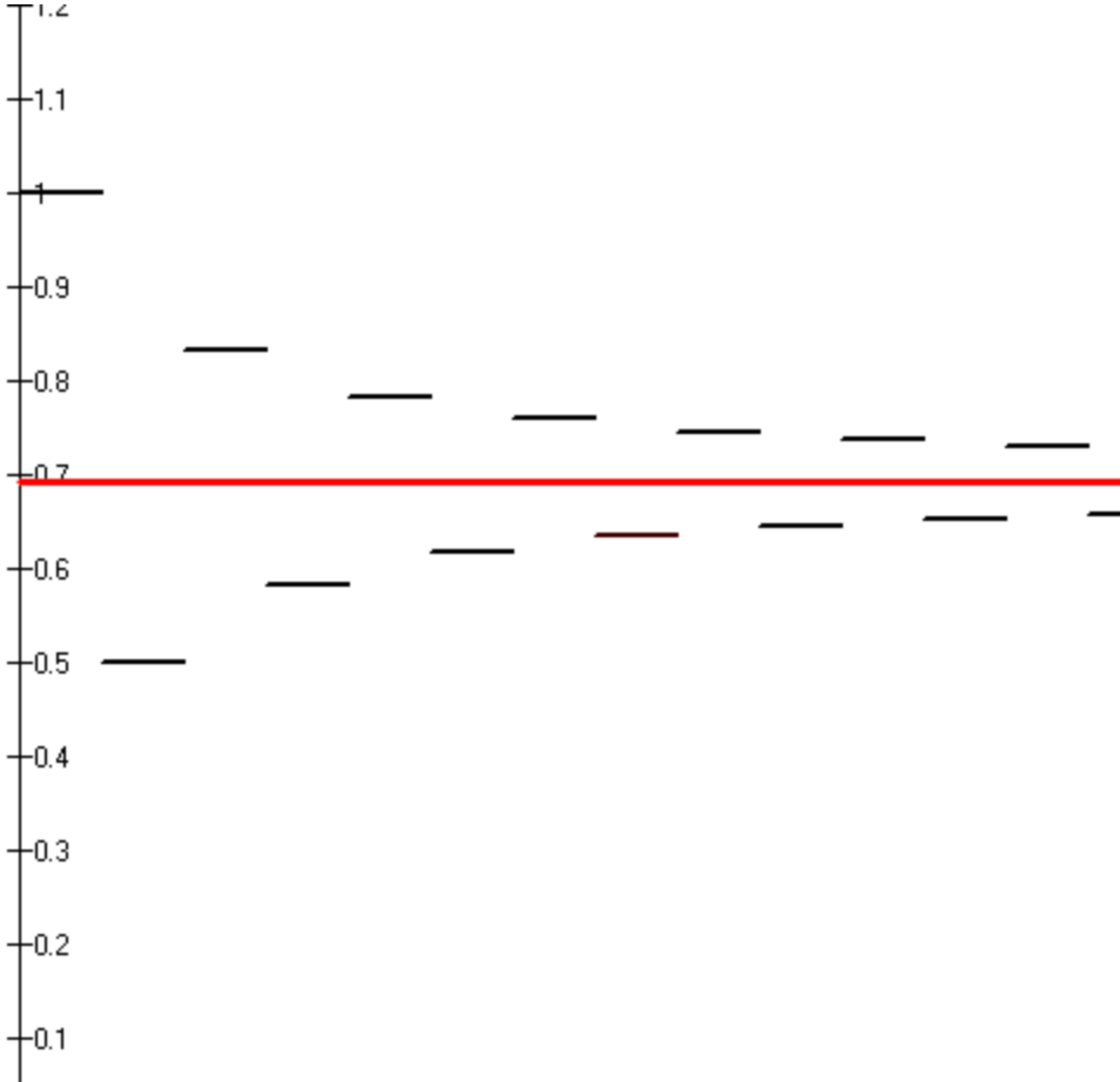
- Absolutely v.s. conditionally convergent sums

$$S = \sum_{i=1}^{\infty} a_i \qquad \sum_{i=1}^{\infty} |a_i| < \infty$$

- Conditional series depend on the ordering

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots = \frac{3}{2} \ln 2$$



Oscillatory integrals

Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

- Absolutely v.s. conditionally convergent integrals

$$I = \int f(x) \mathrm{d}x \qquad \int |f(x)| \mathrm{d}x < \infty$$

$$\boxed{\int_{-\infty}^{\infty} e^{if(x)} \mathrm{d}x \quad \text{or} \quad \int_{x(0)=x_0}^{x(1)=x_1} e^{iS[x(t)]} \mathscr{D}x(t)}$$

- Fubini's theorem

$$\iint f(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int \left[\int f(x,y)\, \mathrm{d}x \right] \mathrm{d}y = \int \left[\int f(x,y)\, \mathrm{d}y \right] \mathrm{d}x$$

- Dominated convergence theorem

$$\lim_{n\to\infty} \left[\int f_n(x) \mathrm{d}x \right] = \int \left[\lim_{n\to\infty} f_n(x) \right] \mathrm{d}x \quad \text{when} \quad |f_n(x)| \leq g(x) \quad \forall n \quad \text{with} \quad \int |g(x)| \mathrm{d}x < \infty$$

Oscillatory integrals

Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

- Fresnel integral

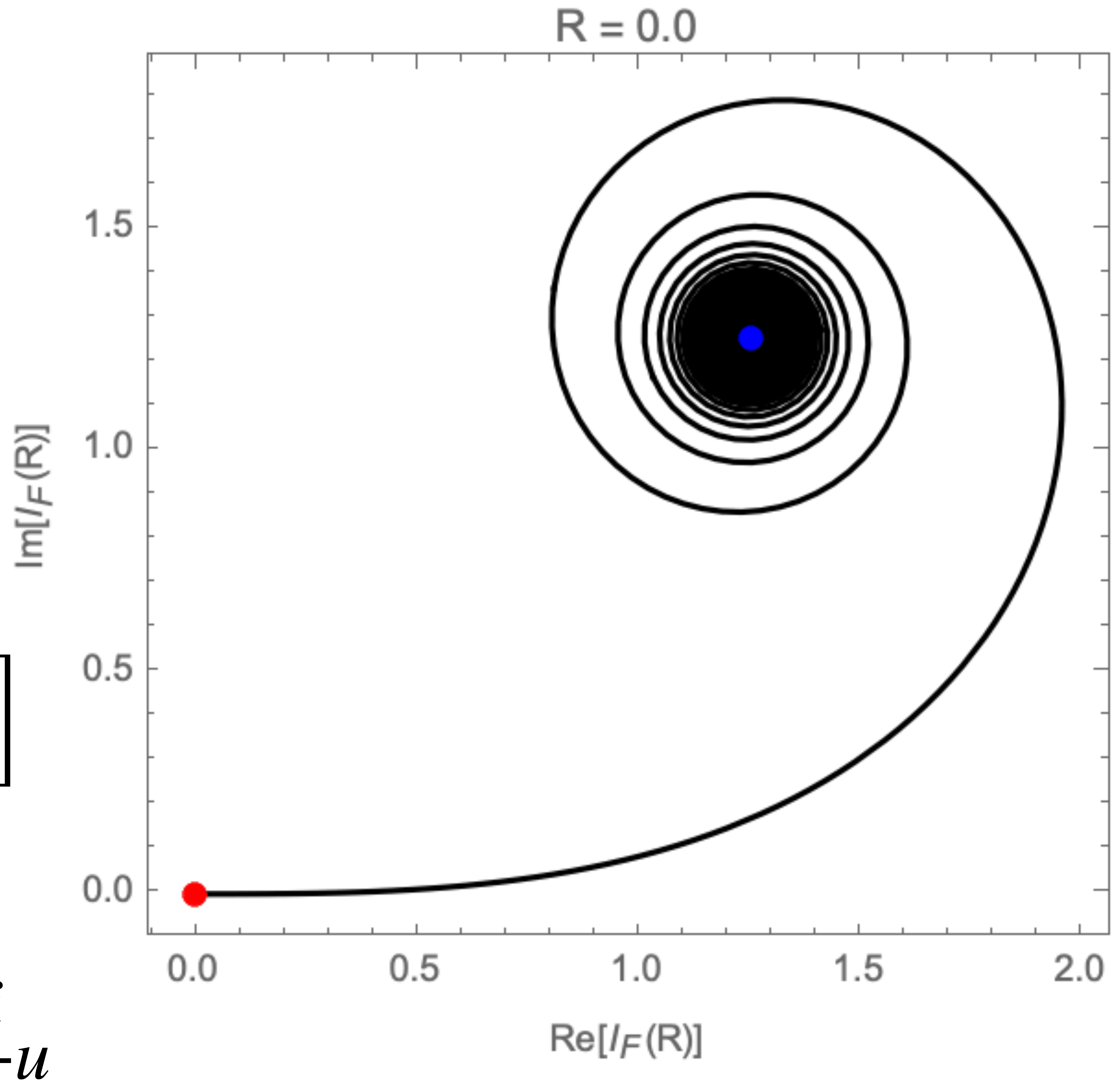
$$\int_{-\infty}^{\infty} e^{ix^2}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} e^{ix^2}\mathrm{d}x = (1+i)\sqrt{\frac{\pi}{2}}$$

- Multi-dimensional extension

$$\iint_{\mathbb{R}^2} e^{i(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \lim_{R\to\infty} 2\pi\int_{0}^{R} r\,e^{ir^2}\mathrm{d}r = \lim_{R\to\infty}\left[i\pi - \pi e^{iR^2}\right]$$

- Complex analysis

$$\int_{-\infty}^{\infty} e^{ix^2}\,\mathrm{d}x = \frac{1+i}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = (1+i)\sqrt{\frac{\pi}{2}} \qquad x = \frac{1+i}{\sqrt{2}}u$$



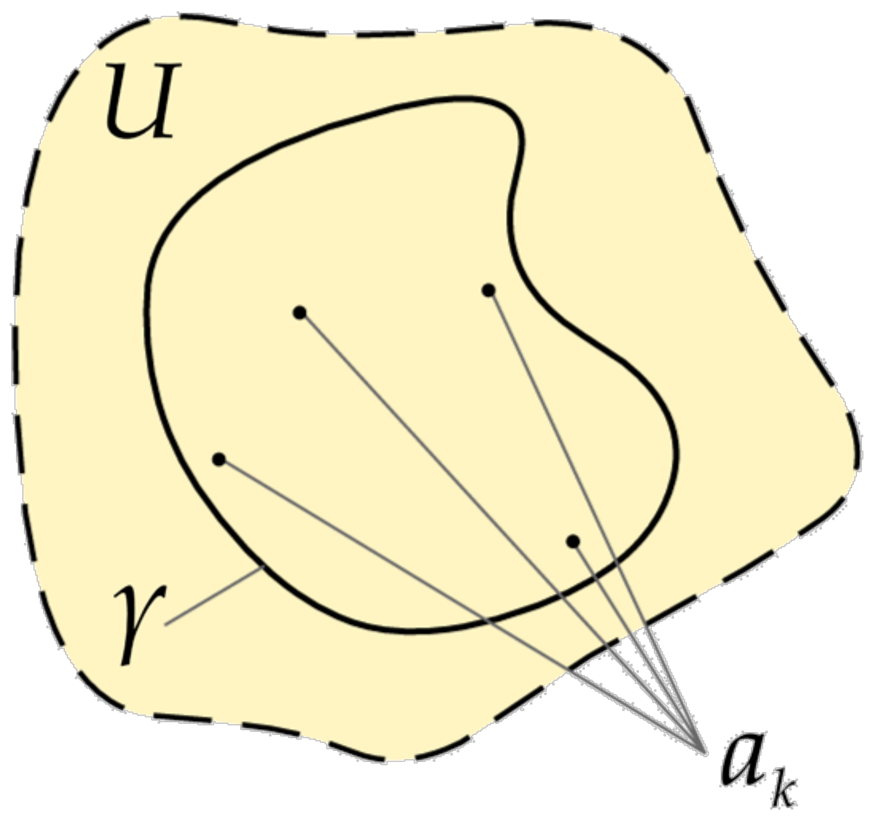
Oscillatory integrals

Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

- Analytic functions

$$f : \mathbb{C} \to \mathbb{C} \qquad f(x+iy) = u(x+iy) + i\,v(x+iy)$$

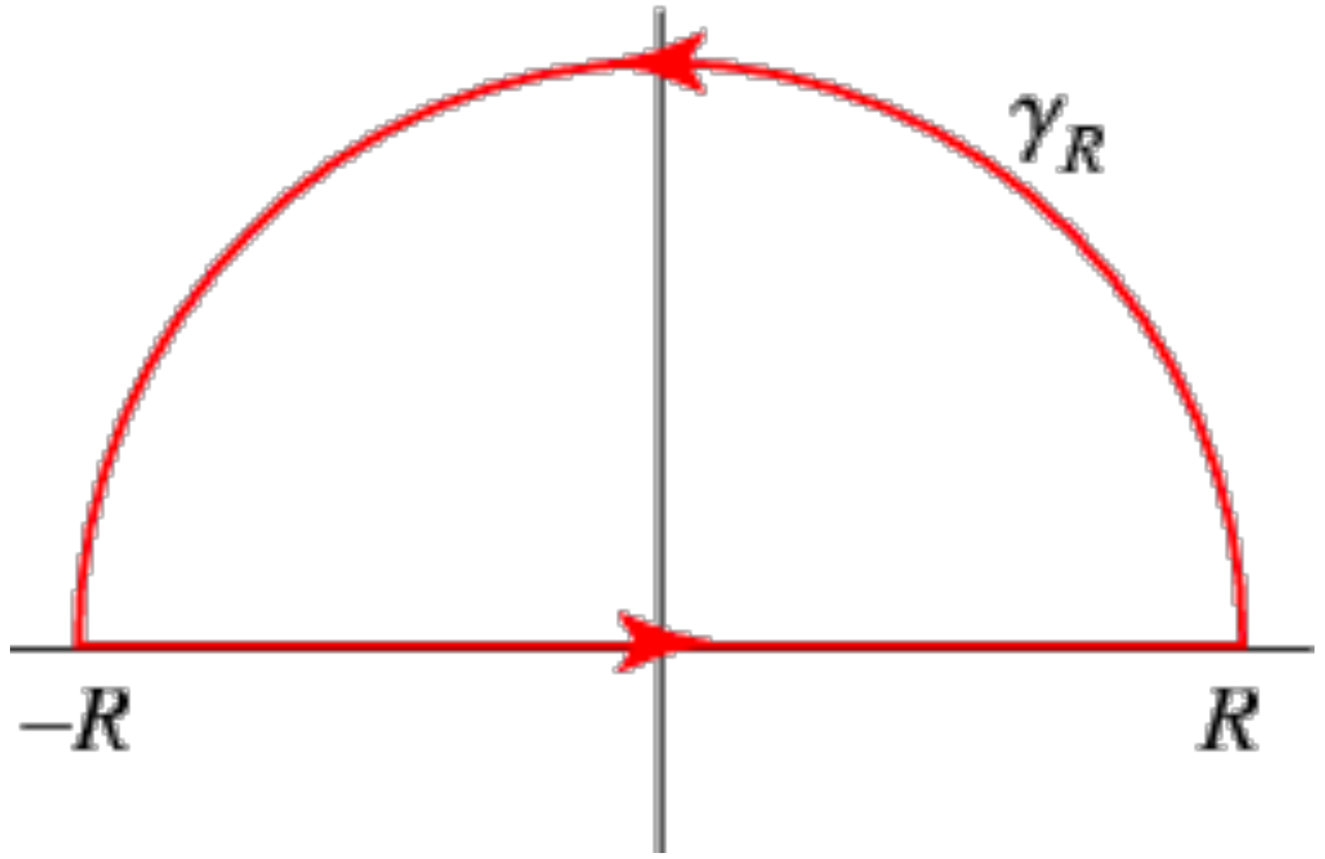
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



- Cauchy's integral theorem

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_k \mathbf{Res}(f, a_k)$$

- Equivalence of integration contours

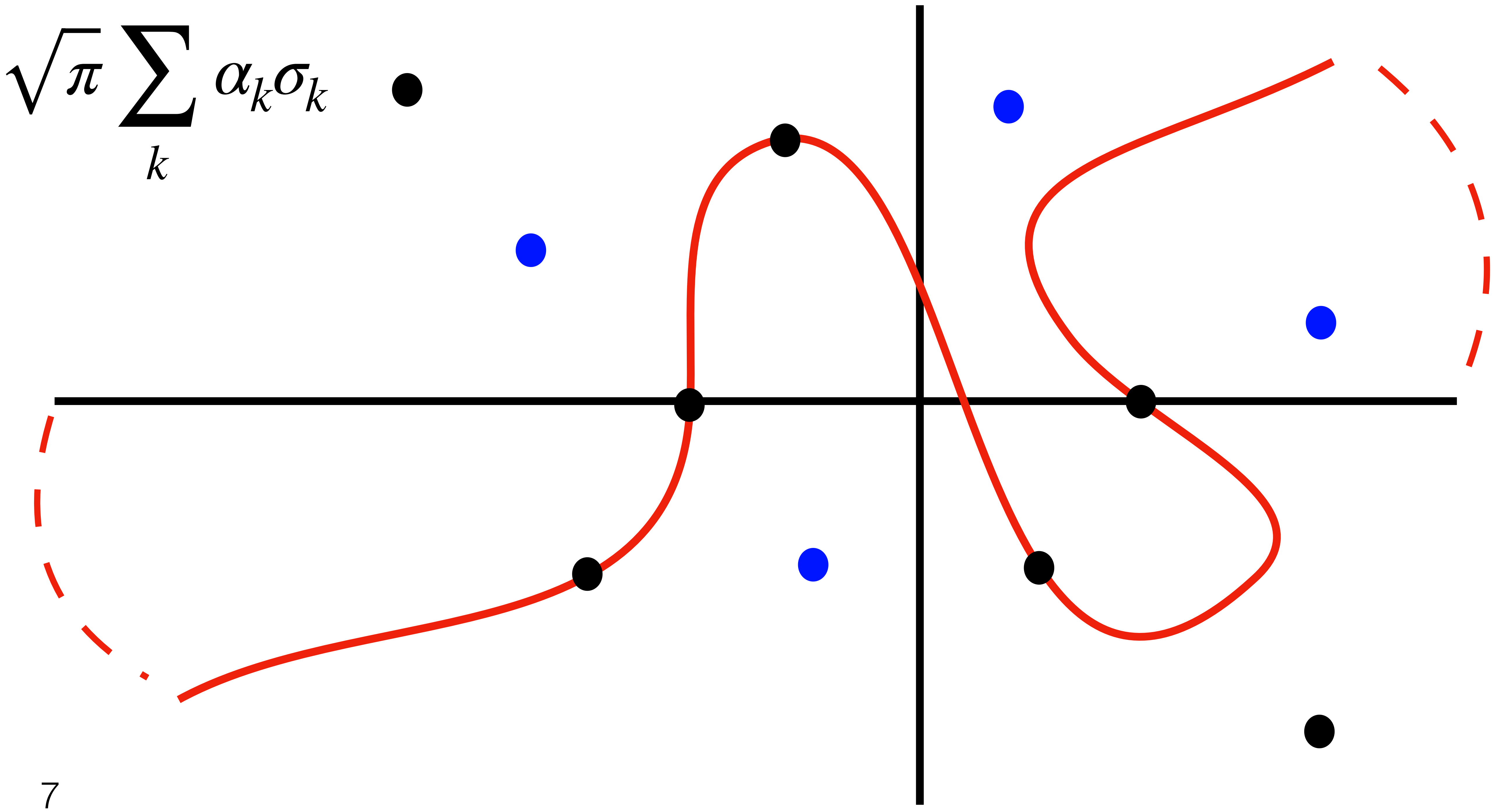


Oscillatory integrals

Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

- Saddle point methods (WKB or Eikonal approximation)

$$\int_{-\infty}^{\infty} e^{if(x)}\,\mathrm{d}x \approx \sum_k \int \alpha_k \exp\left[-x^2/\sigma_k^2\right]\mathrm{d}x = \sqrt{\pi}\sum_k \alpha_k \sigma_k$$



- What is the optimal contour?
- Which saddle points to include?

Oscillatory integrals

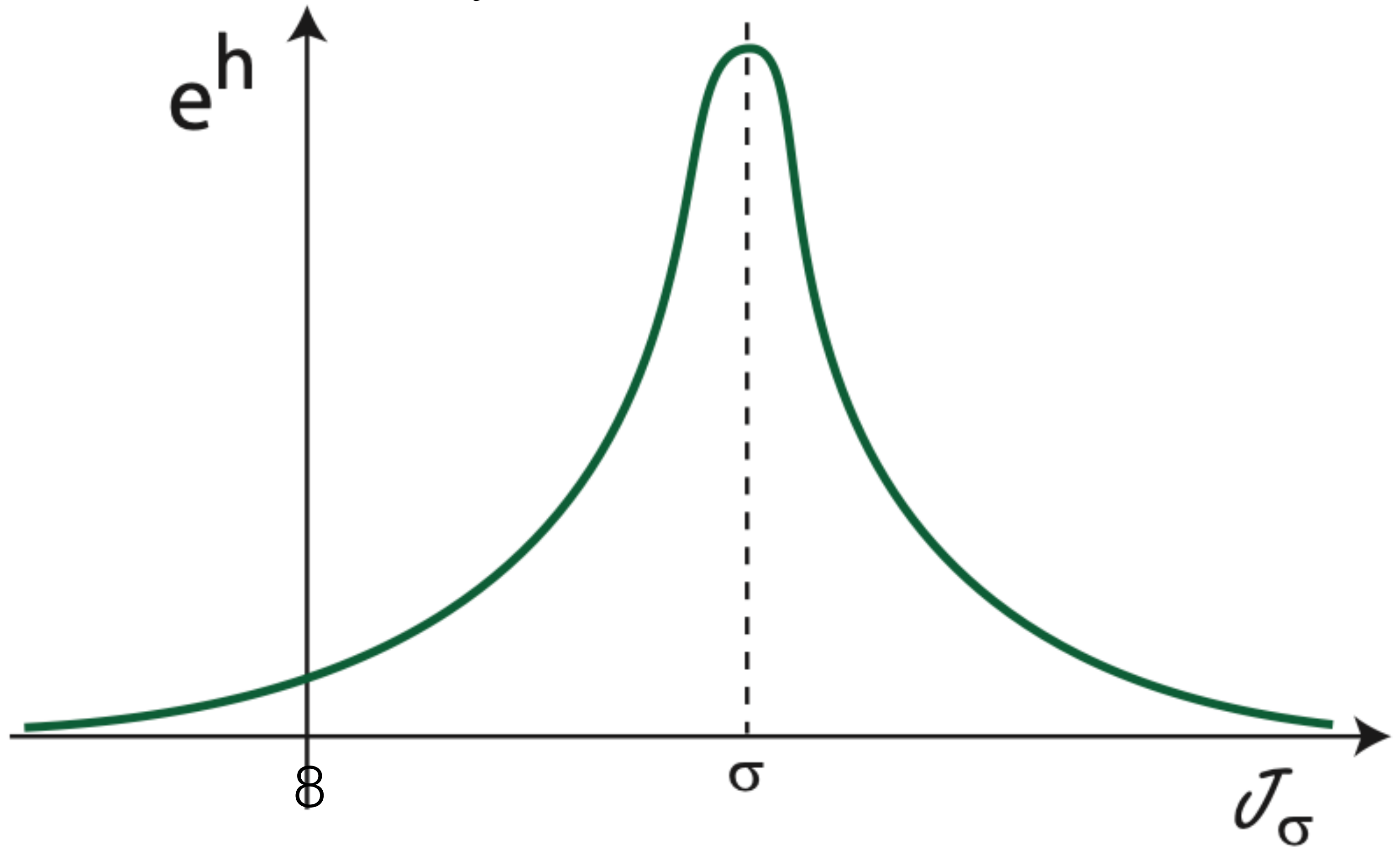
Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

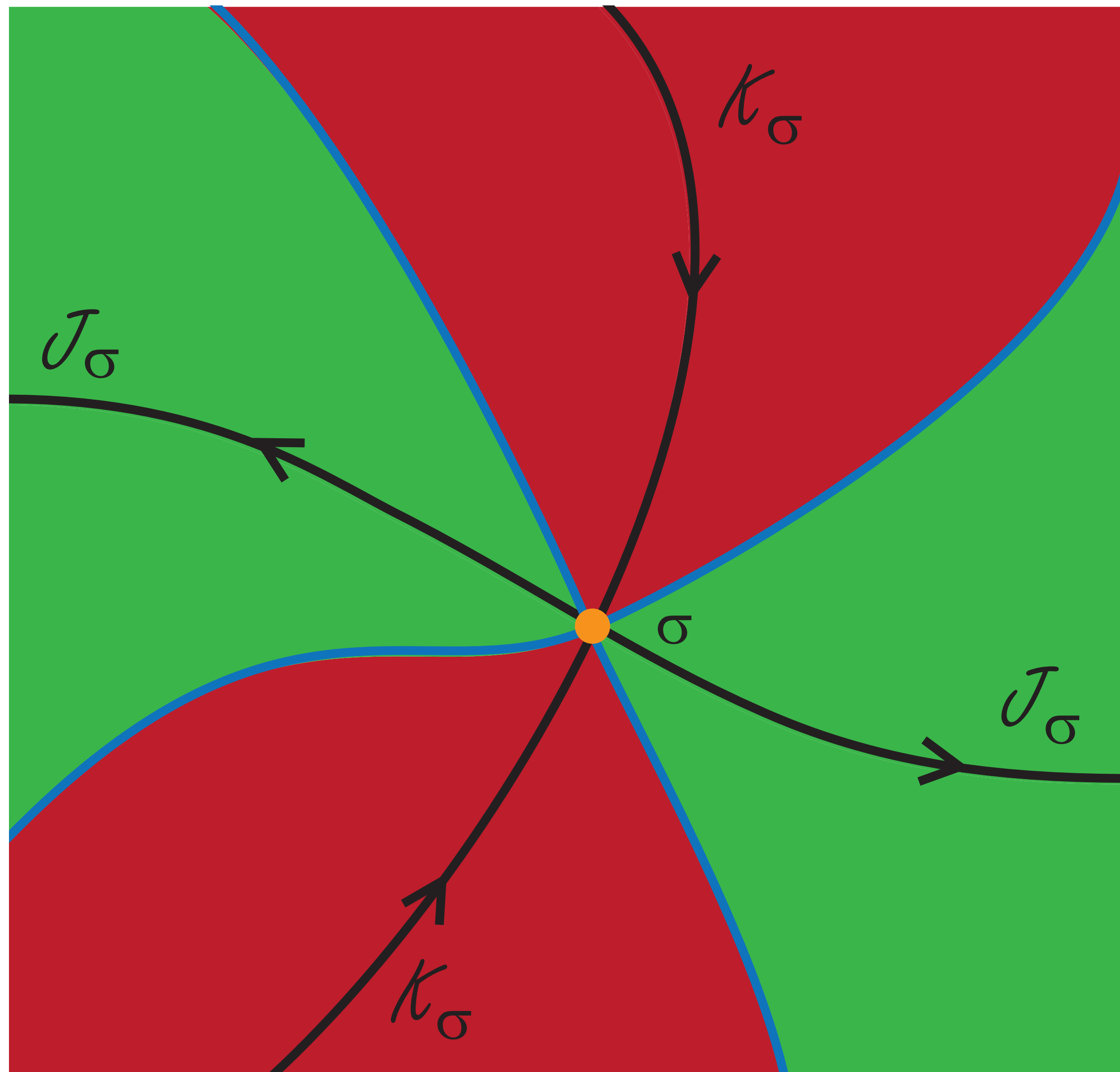
- Picard-Lefschetz theory
  - Find all saddle points
  - Find all steepest ascent/descent contours
  - Intersect the ascent with the original integration domain

$$I = \int_C e^{if(z)}\,\mathrm{d}z \qquad f(z) = h(z) + iH(z)$$

$$f'(z_i) = 0 \qquad \frac{\partial \gamma_\lambda}{\partial \lambda} = -\nabla h(\gamma_\lambda)\,, \quad \gamma_0 = z_i$$

$$I = \sum_i n_i\, e^{iH(z_i)} \int_{\mathcal{J}_i} e^{h(z)}\,\mathrm{d}z$$





Oscillatory integrals


Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

- Defining conditional integrals
  - Use an analytic absolutely convergent regulator which vanishes in the limit

    $$\int |\, g_R(x)\,|\, \mathrm{d}x < \infty, \quad R < \infty \qquad \lim_{R\to\infty} g_R(x) = 1$$

  - For example

    $$g_R(x) = e^{-x^2/R^2}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{if(x)}\, \mathrm{d}x &\equiv \lim_{R\to\infty} \int_{-\infty}^{\infty} e^{if(z)} g_R(x)\, \mathrm{d}x \\
&= \lim_{R\to\infty} \int_{\mathcal{J}} e^{if(z)} g_R(x)\, \mathrm{d}x \\
&= \int_{\mathcal{J}} e^{if(z)} \left[\lim_{R\to\infty} g_R(x) \right] \mathrm{d}x \\
&= \int_{\mathcal{J}} e^{if(z)}\, \mathrm{d}x
\end{aligned}$$

$$\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 + \ldots$$

Oscillatory integrals

Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

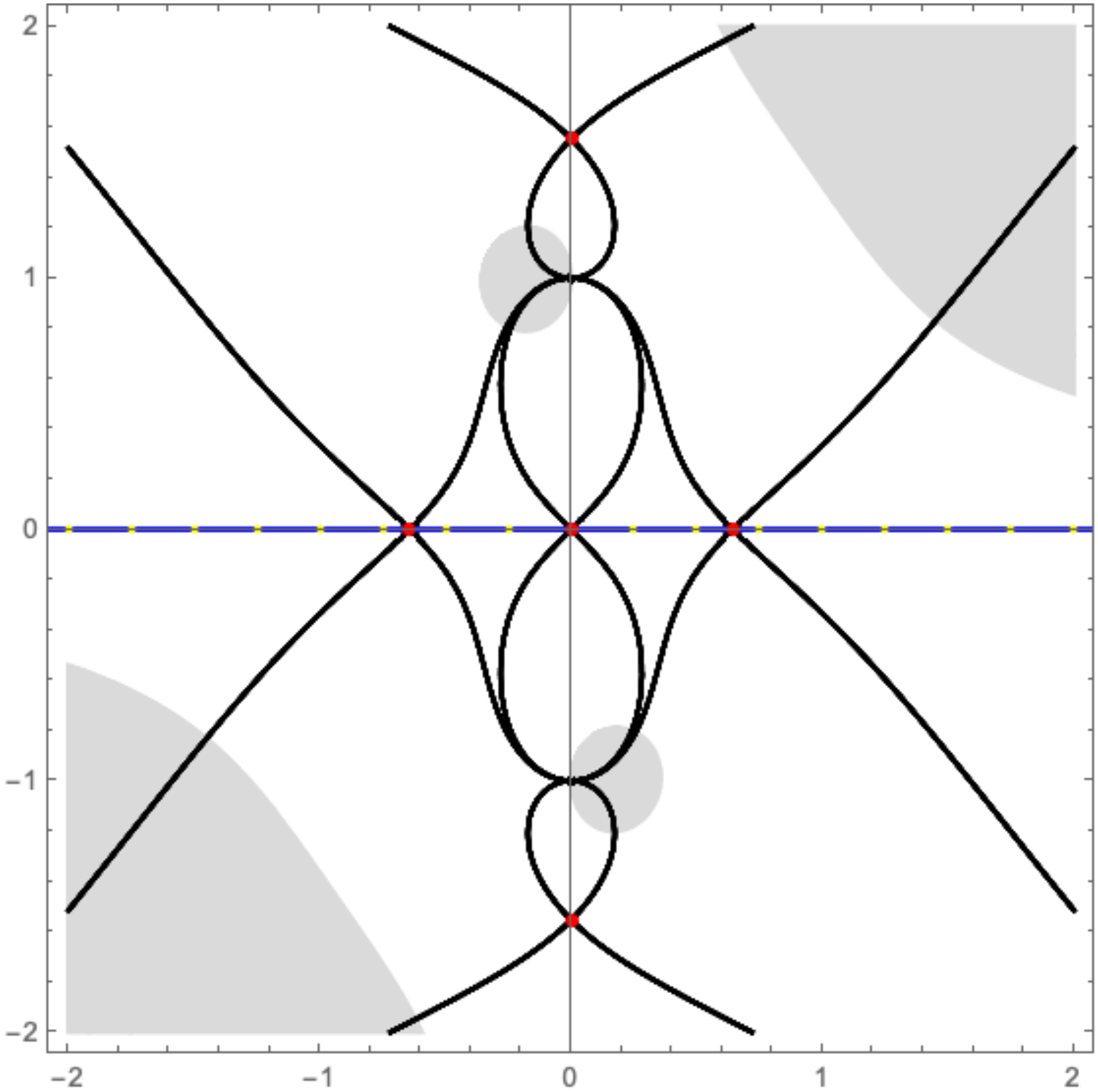
- Intersection method

$$I = \sum_i n_i e^{iH(z)} \int_{\mathcal{J}_i} e^{h(z)}\, \mathrm{d}z \qquad n_i = \langle C, \mathcal{K}_i \rangle$$

- Flow method

$$I = \int_C e^{if(z)} \mathrm{d}z = \int_{\gamma_\lambda(C)} e^{if(z)} \mathrm{d}z$$

$$\lim_{\lambda \to \infty} C_\lambda = \mathcal{J}$$



website: p-lpi.github.io

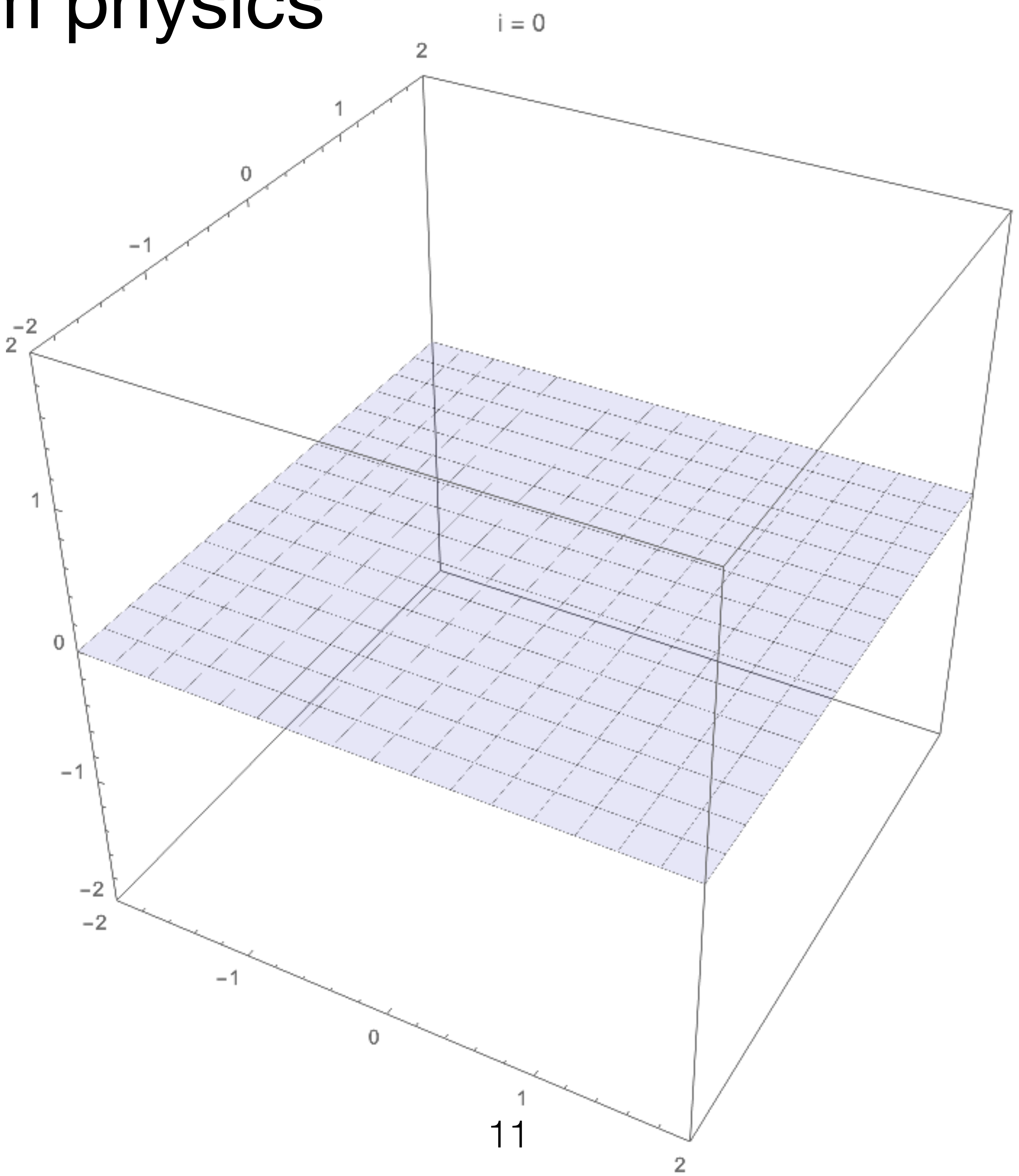
Oscillatory integrals

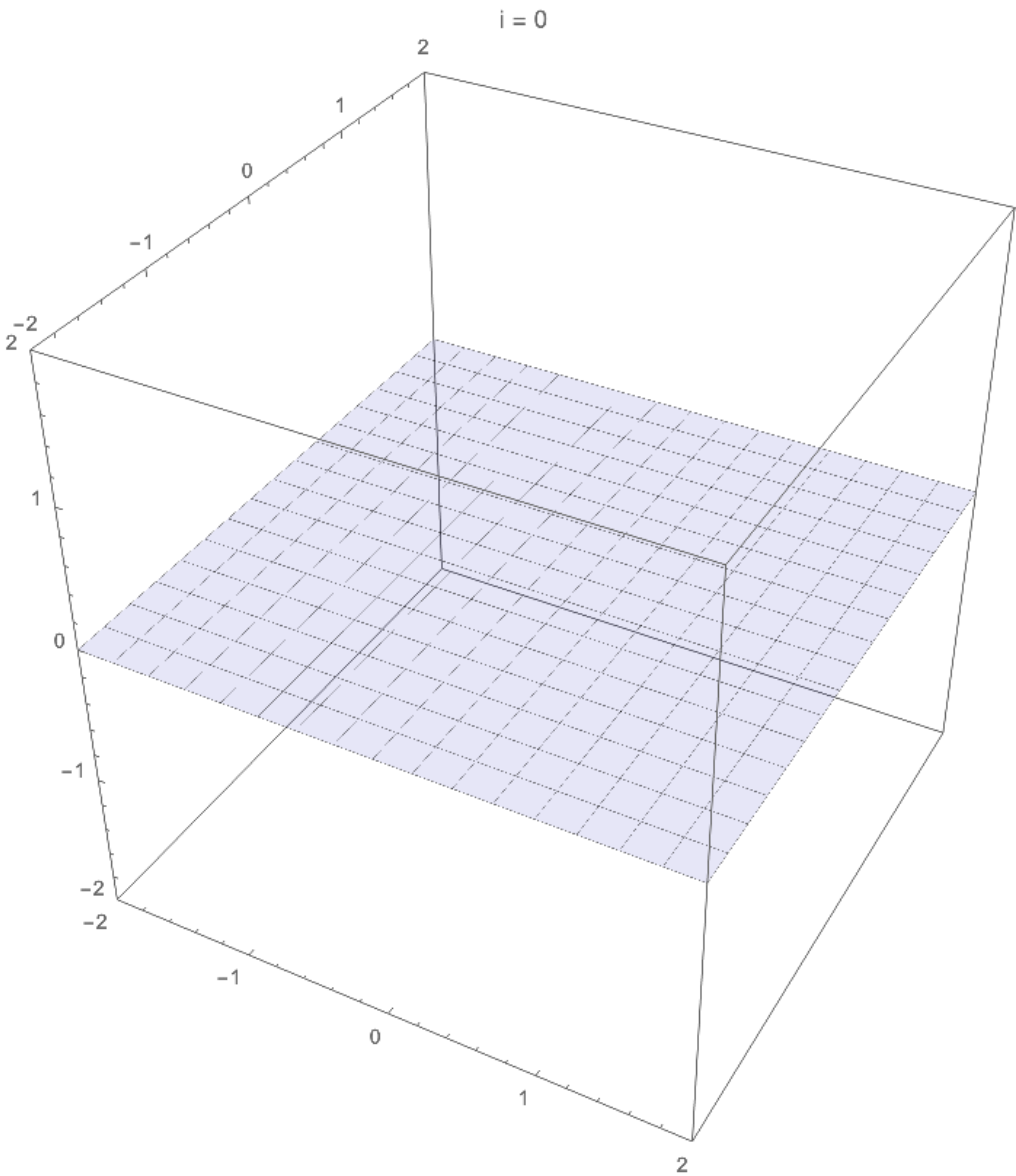
Oscillatory integrals occur in may places, ranging from classical systems, wave optics, to quantum physics

- Flow method

$$I = \int_{\mathbb{R}^2} e^{i\nu(x^2+\phi(x))} \mathrm{d}^2 x$$

$$\phi(x) = \frac{1}{1 + x_1^2 + 2x_2^2}$$





website: p-lpi.github.io

Single plane lensing

Kirchhoff-Fresnel integral

$$\int_{x(0)=x_s}^{x(1)=x_{obs}} e^{iS[x]}\,\mathscr{D}x = \int e^{i\frac{\omega}{2c}\left[\frac{(x_\perp-\mu)^2}{d} - \int \frac{\omega_p^2(x_\perp,z)}{\omega^2}\mathrm{d}z\right]}\mathrm{d}^D x_\perp$$

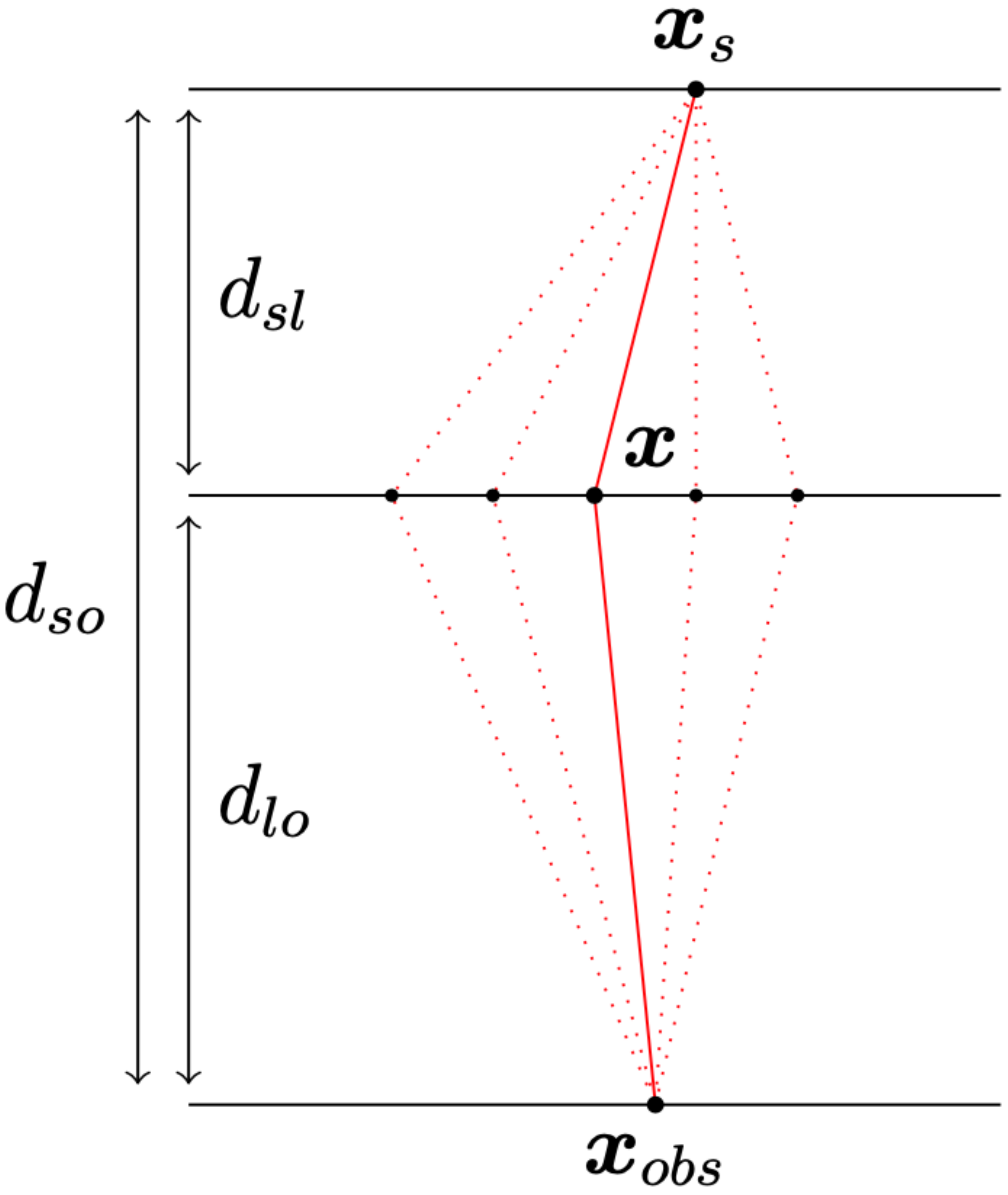
$$\frac{1}{d} = \frac{1}{d_{sl}} + \frac{1}{d_{lo}} \qquad \omega_p^2 = \frac{n_e(x)e^2}{\epsilon_0 m_e}$$

which takes the dimensionless form

$$\Psi(\mu,\nu) = \left(\frac{\nu}{\pi}\right)^{D/2} \int e^{i\nu\left[\frac{1}{2}(x-\mu)^2 + \varphi(x)\right]}\,\mathrm{d}^D x$$

with the time delay

$$\Phi(x) = \frac{1}{2}(x-\mu)^2 + \varphi(x)$$



Single plane lensing

Multi-image regions separated by caustics

$$\Phi(x) = \frac{1}{2}(x-\mu)^2 + \varphi(x)$$

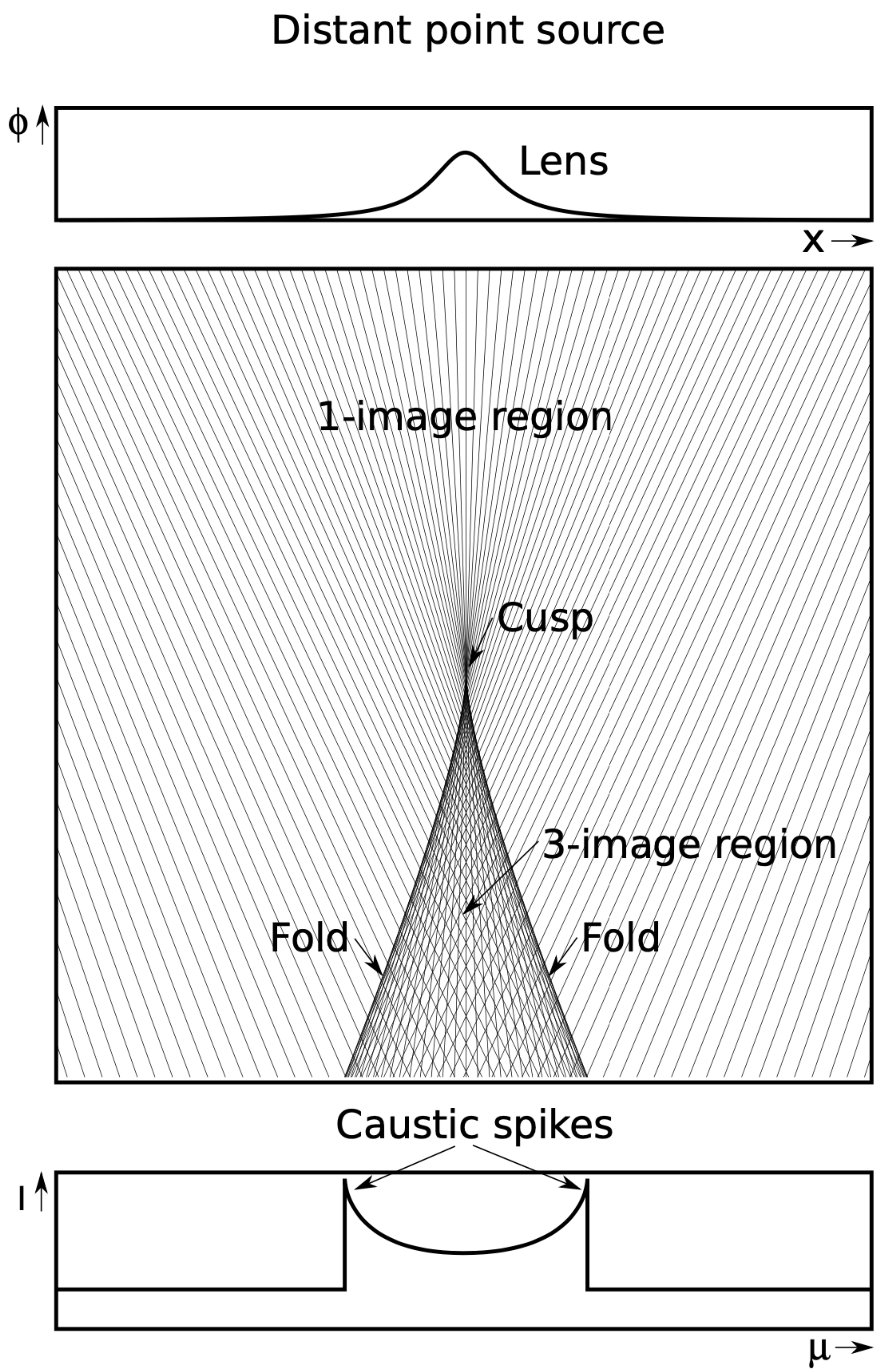
In geometric optics Fermat's principle gives the Lagrangian map

$$\nabla\Phi(x) = 0$$

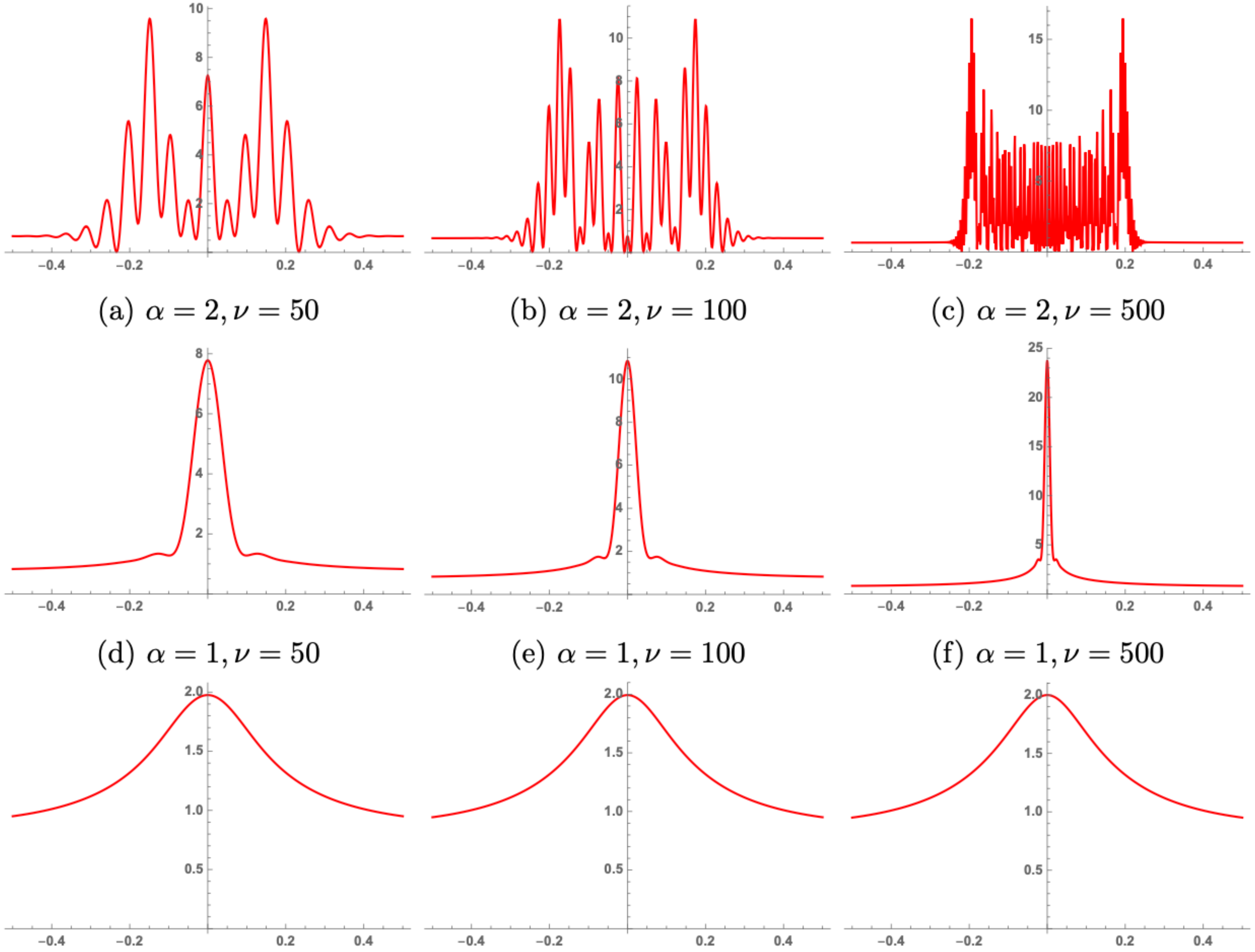
$$\xi(x) = x + \alpha(x) \qquad \alpha(x) = \nabla\varphi(x)$$

with the intensity field

$$I(\mu) = \sum_{x\in\xi^{-1}(\mu)} \frac{1}{|\det\nabla\xi(x)|}$$



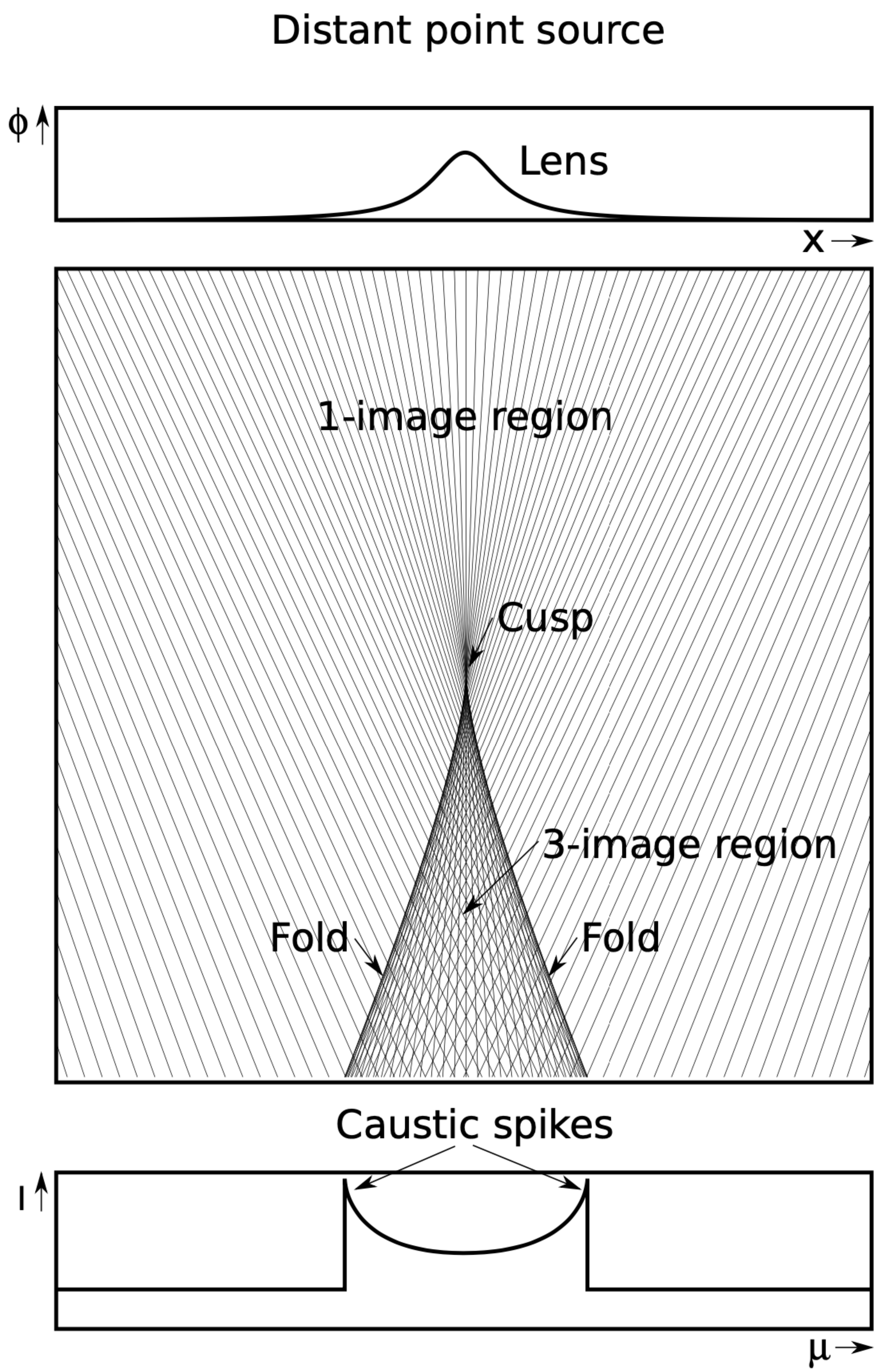
Single plane lensing



(a) $\alpha = 2, \nu = 50$ (b) $\alpha = 2, \nu = 100$ (c) $\alpha = 2, \nu = 500$

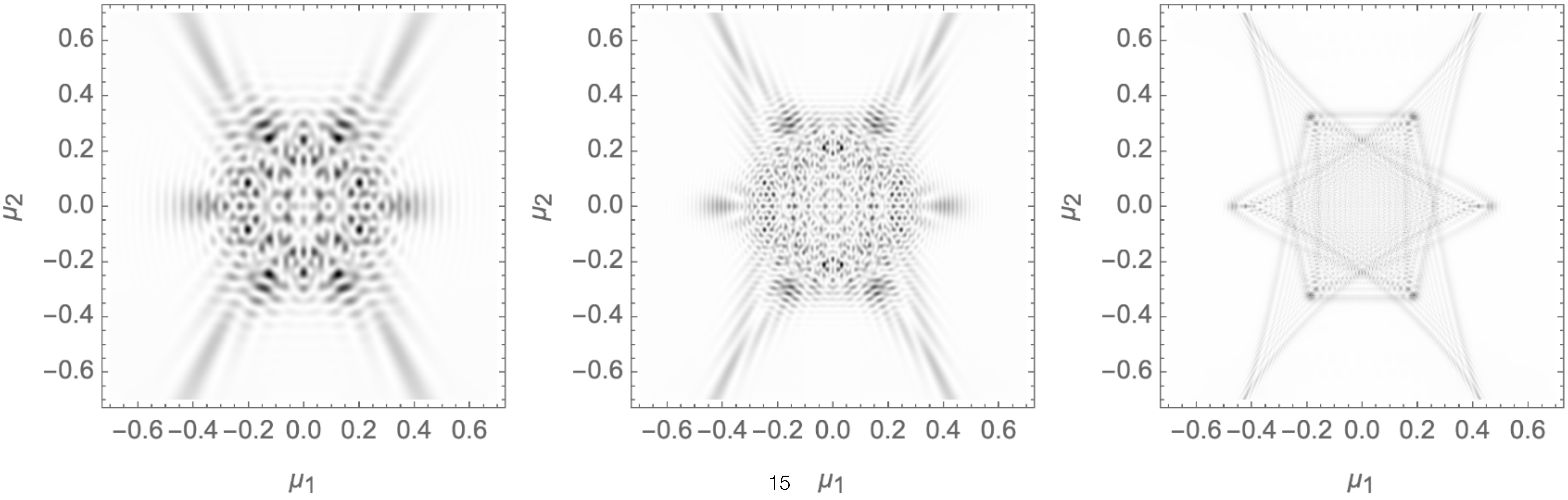
(d) $\alpha = 1, \nu = 50$ (e) $\alpha = 1, \nu = 100$ (f) $\alpha = 1, \nu = 500$

(g) $\alpha = 1/2, \nu = 50$ (h) $\alpha = 1/2, \nu = 100$ (i) $\alpha = 1/2, \nu = 500$



Single plane lensing

$$I = \int_{\mathbb{R}^2} e^{i\nu\left[\frac{1}{2}(x-\mu)^2 + \varphi(x)\right]} \mathrm{d}^2 x \qquad \varphi(x) = \frac{1}{1 + x_1^4 + x_2^2}$$



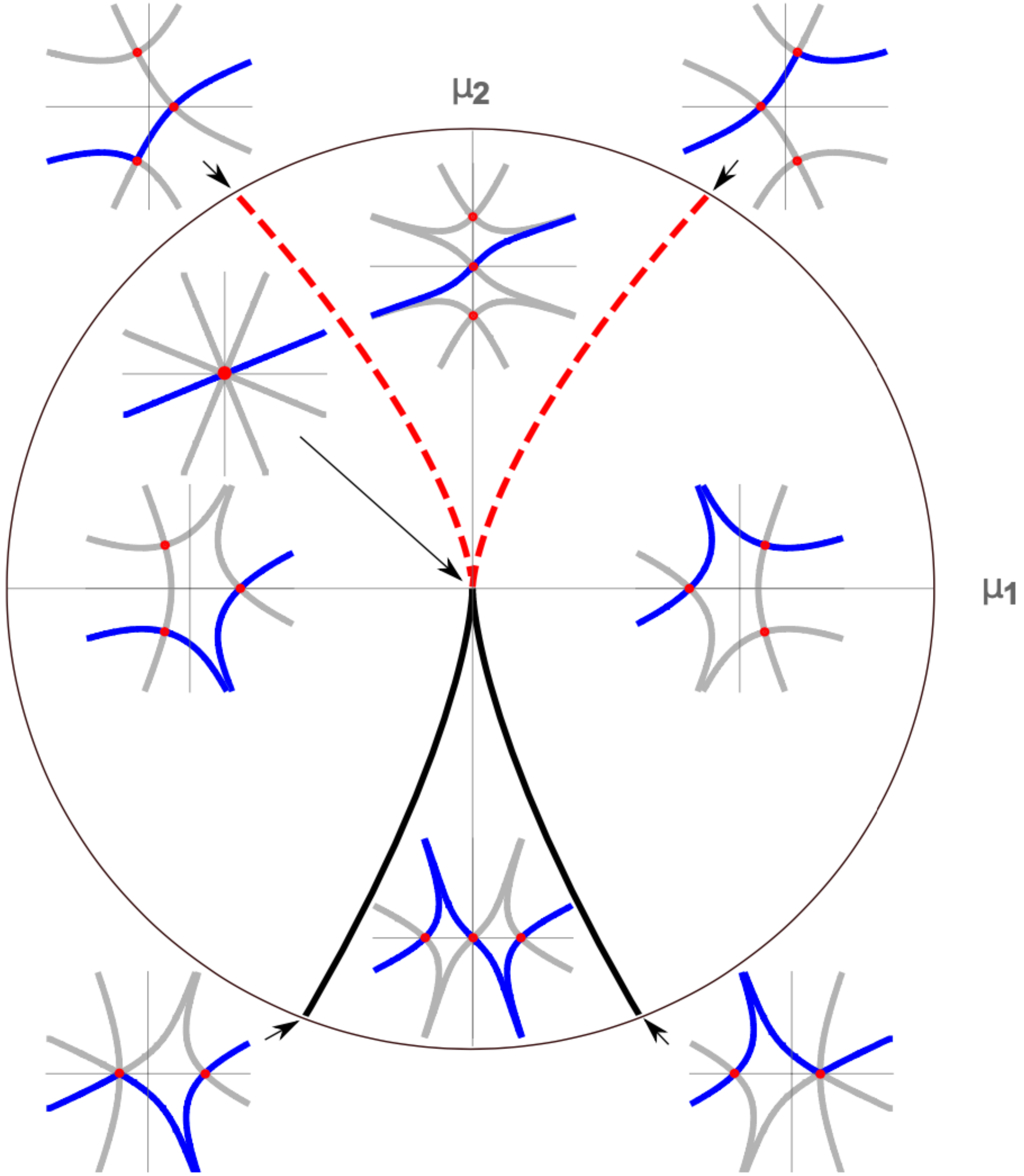
Single plane lensing

$$\Psi(\mu,\nu) = \left(\frac{\nu}{\pi}\right)^{N/2} \int e^{i\nu\phi(x,\mu)} \mathrm{d}x$$

Name	Symbol	K	N	$\phi(\boldsymbol{x};\boldsymbol{\mu})$
Maximum/minimum	$A_1^{\pm}$	0	1	$\pm x^2$
Fold	A_2	1	1	$x^3/3 + \mu x$
Cusp	A_3	2	1	$x^4/4 + \mu_2 x^2/2 + \mu_1 x$
Swallowtail	A_4	3	1	$x^5/5 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Butterfly	A_5	4	1	$x^6/6 + \mu_4 x^4/4 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Elliptic umbilic	D_4^-	3	2	$x_1^3 - 3x_1 x_2^2 - \mu_3(x_1^2 + x_2^2) - \mu_2 x_2 - \mu_1 x_1$
Hyperbolic umbilic	D_4^+	3	2	$x_1^3 + x_2^3 - \mu_3 x_1 x_2 - \mu_2 x_2 - \mu_1 x_1$
Parabolic umbilic	D_5	4	2	$x_1^4 + x_1 x_2^2 + \mu_4 x_2^2 + \mu_3 x_1^2 + \mu_2 x_2 + \mu_1 x_1$

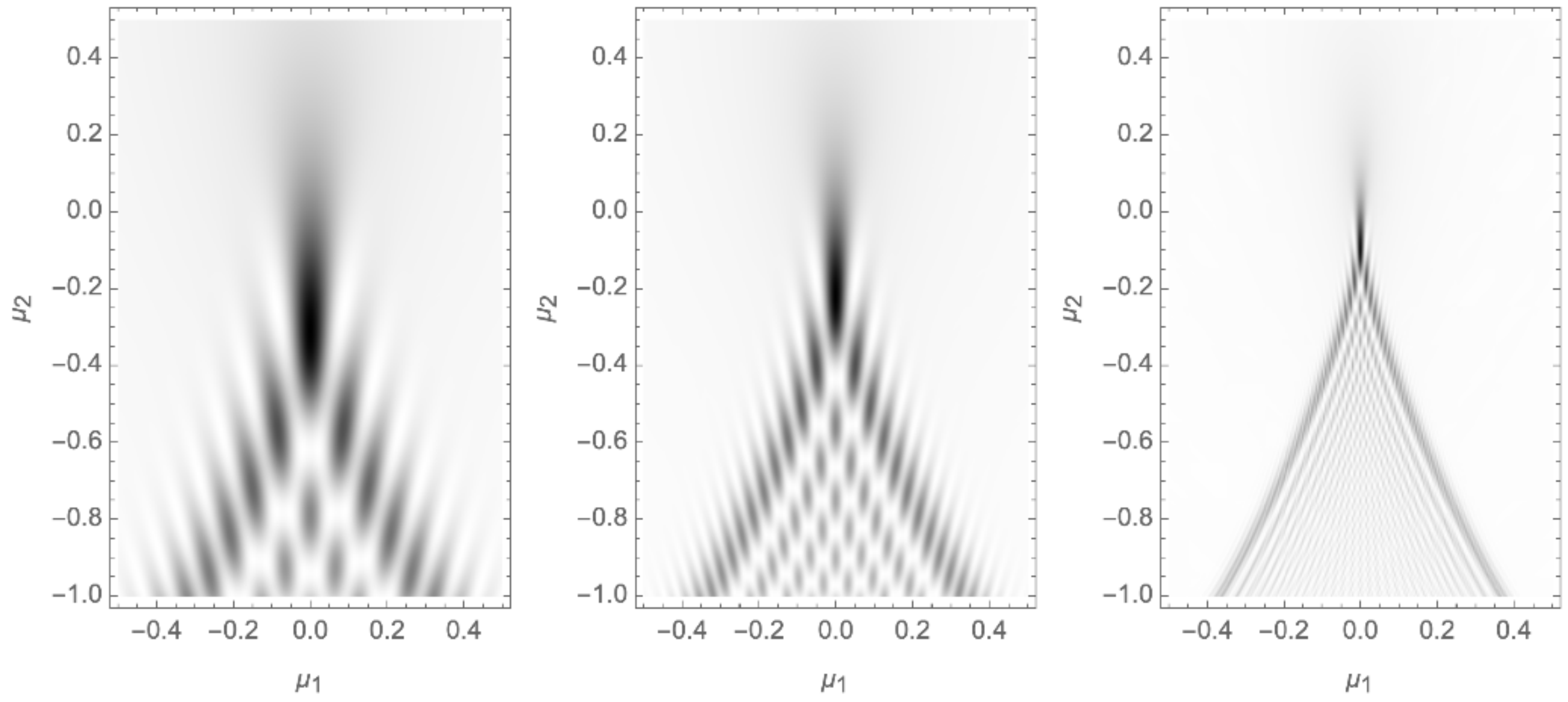
Table I: The unfoldings of the seven elementary catastrophes with codimension $K \leq 4$, with $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_K)$. The normal forms are defined as the unfolding at parameter $\boldsymbol{\mu} = \boldsymbol{0}$, *i.e.*, $\phi(\boldsymbol{x}; \boldsymbol{0})$.

Single plane lensing



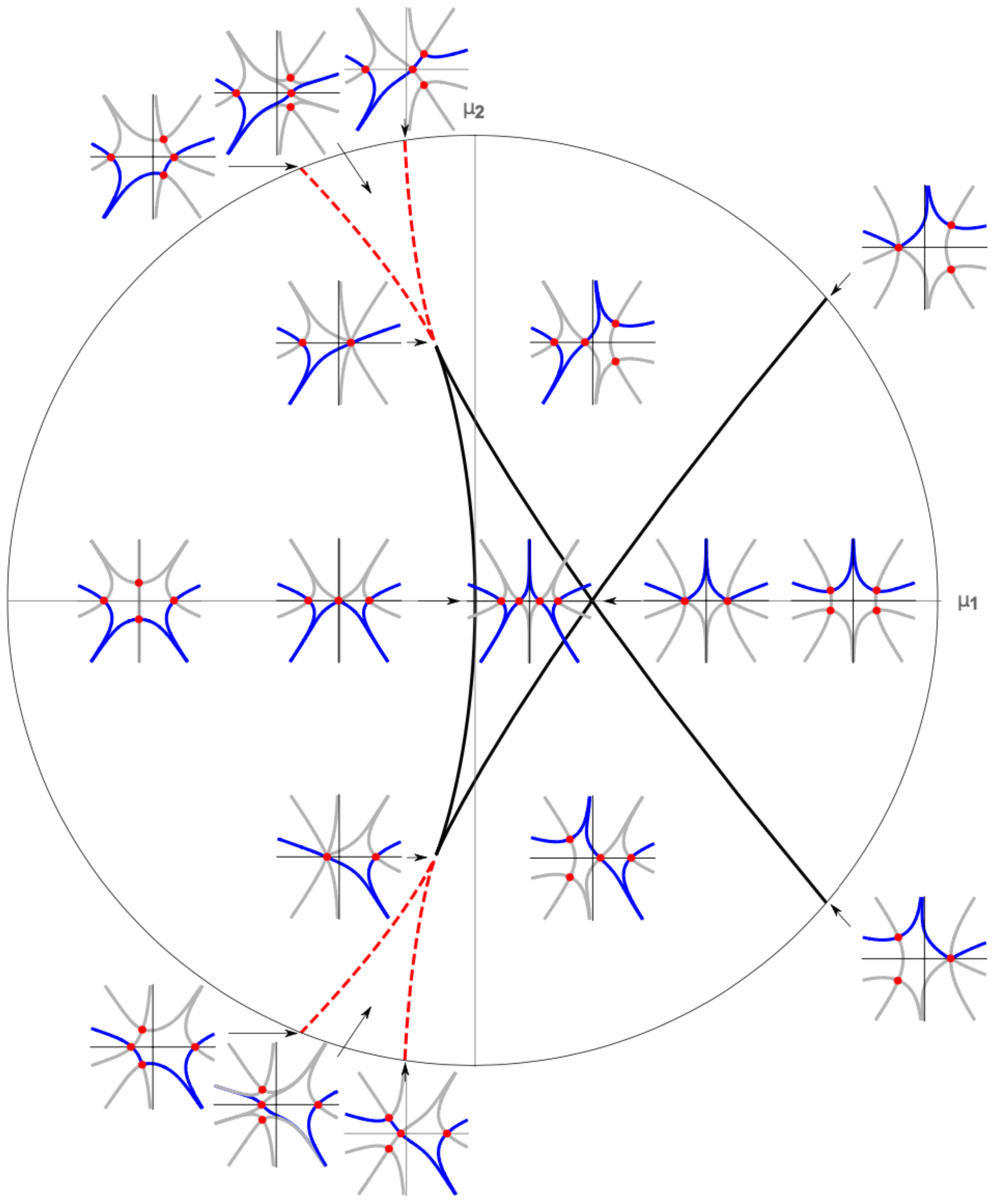
Caustics and Stoke's lines

$$\Psi(\mu,\nu) = \int e^{i\nu(x^4/4+\mu_2 x^2/2+\mu_1 x)}\mathrm{d}x$$



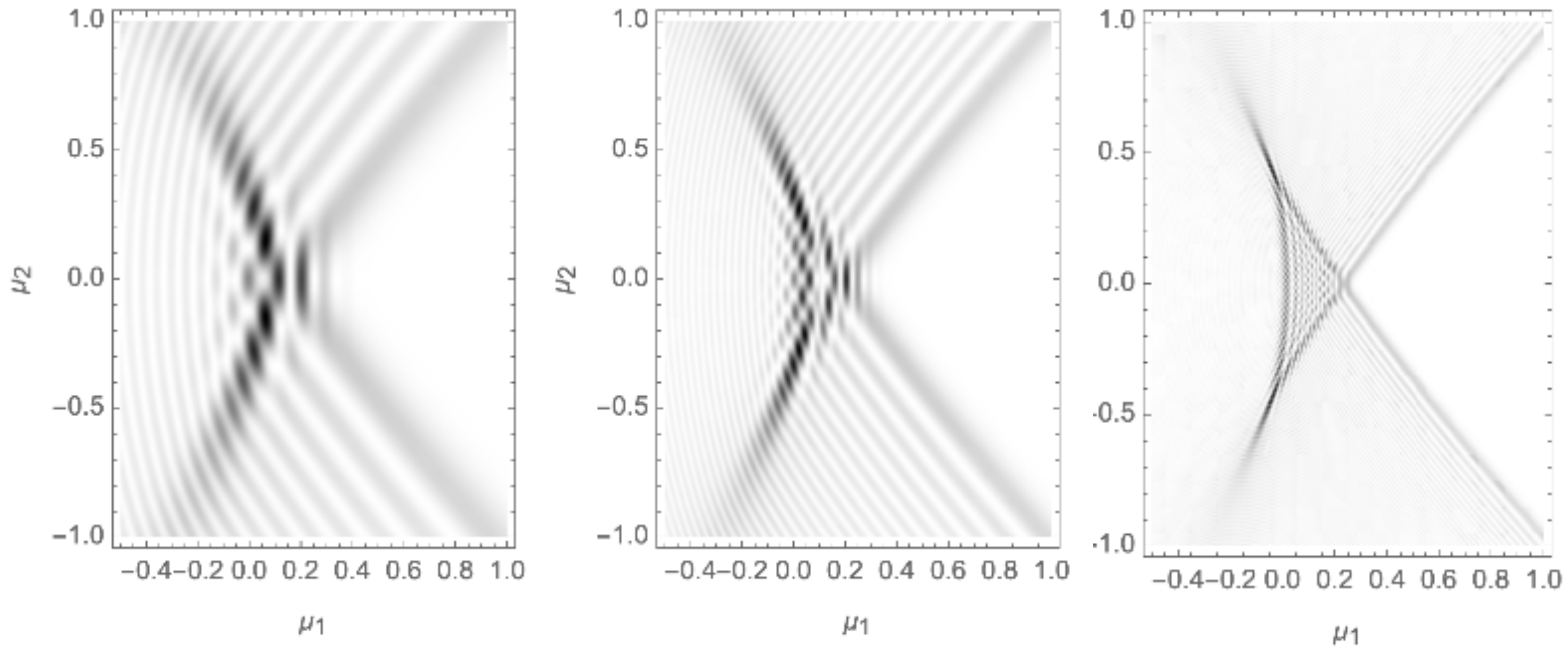
(a) $\nu = 50$ (b) $\nu = 100$ (c) $\nu = 500$

Single plane lensing



Caustics and Stoke's lines

$$\Psi(\mu, \nu) = \int e^{i\nu(x^5/5+\mu_3 x^3/3+\mu_2 x^2/2+\mu_1 x)}\mathrm{d}x$$



Gravitational lensing

In single-plane gravitational lensing, the time delay can be singular

$$\Psi(\mu,\nu) = \int e^{i\nu\left[\frac{1}{2}(x-\mu)^2 + \varphi(x)\right]}\mathrm{d}x \qquad \varphi(x) = -\sum_i f_i \log|x - x_i|$$

with the dimensionless frequency and the mass fraction

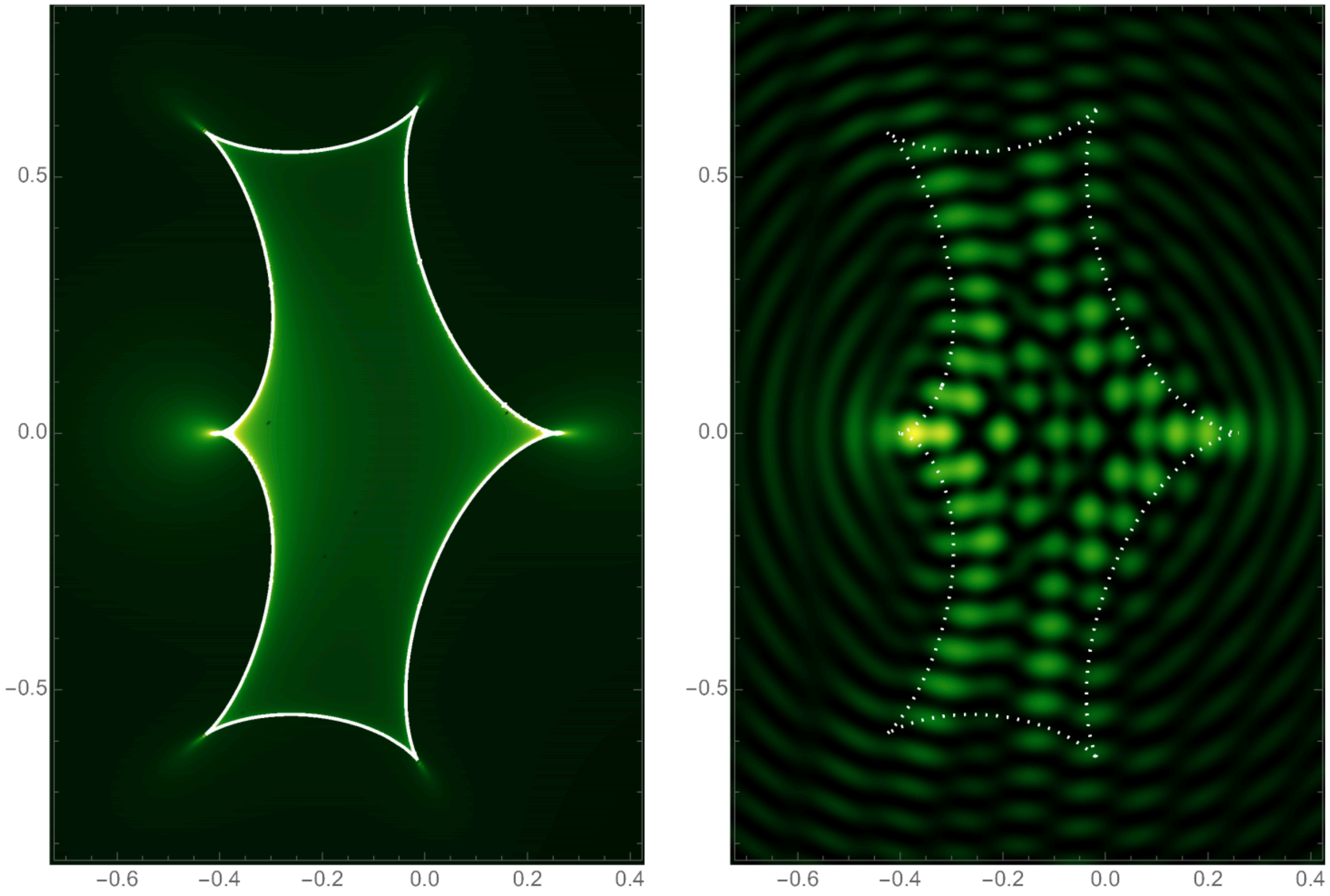
$$\nu = 4GM\omega \qquad f_i = \frac{M_i}{\sum_i M_i}$$

geometric optics is insensitive to the total mass of the gravitational lens.
The binary gravitational lens

$$\Psi(\mu,\nu) = \int e^{i\nu\left[\frac{1}{2}(x-\mu)^2 - f_1\log|x-x_1| - f_2\log|x-x_2|\right]}\mathrm{d}x$$

Gravitational lensing

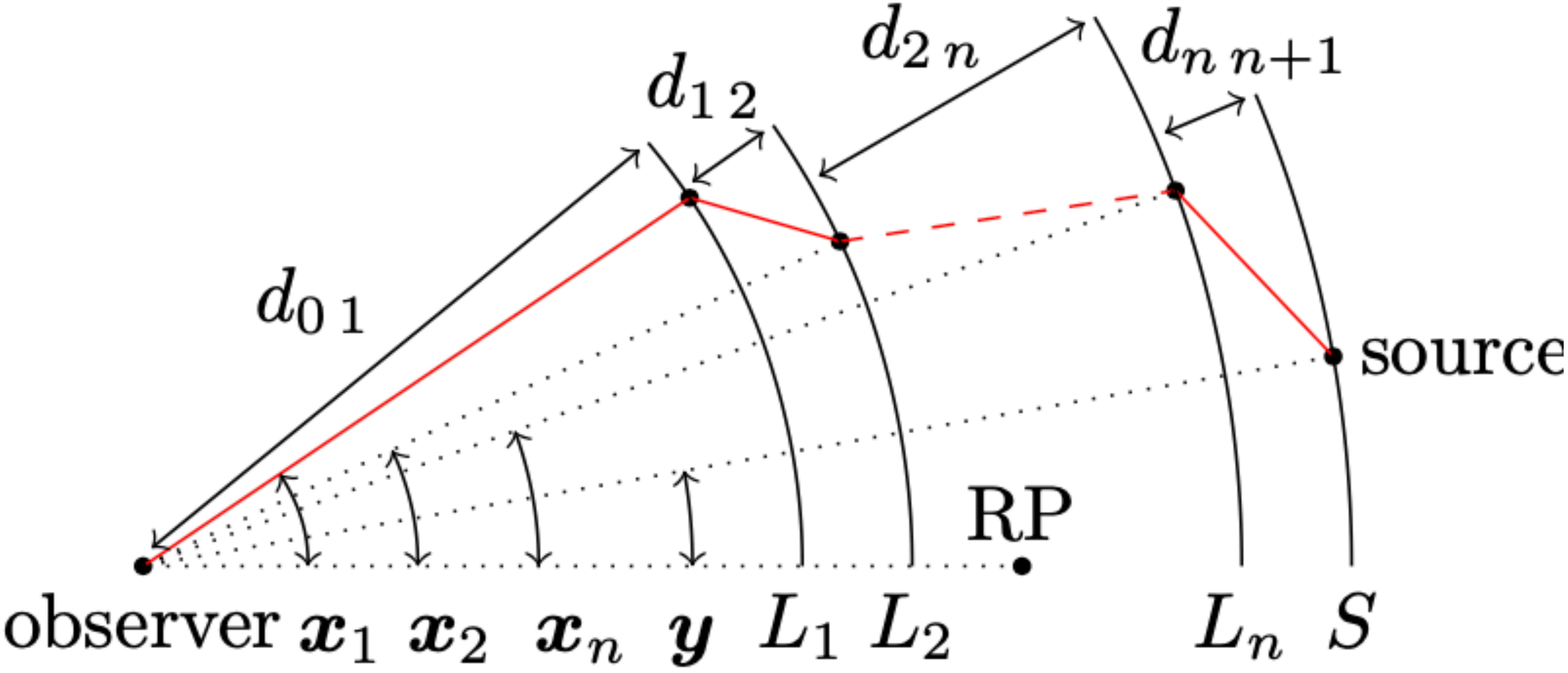
Lensing in in wave optics is much richer than in geometric optics. For gravitational lensing, wave optics breaks the degeneracy and allows to measure the mass. In the future observable with Fast Radio Bursts



(a) ray optics (b) wave optics ($\Omega = 50$)

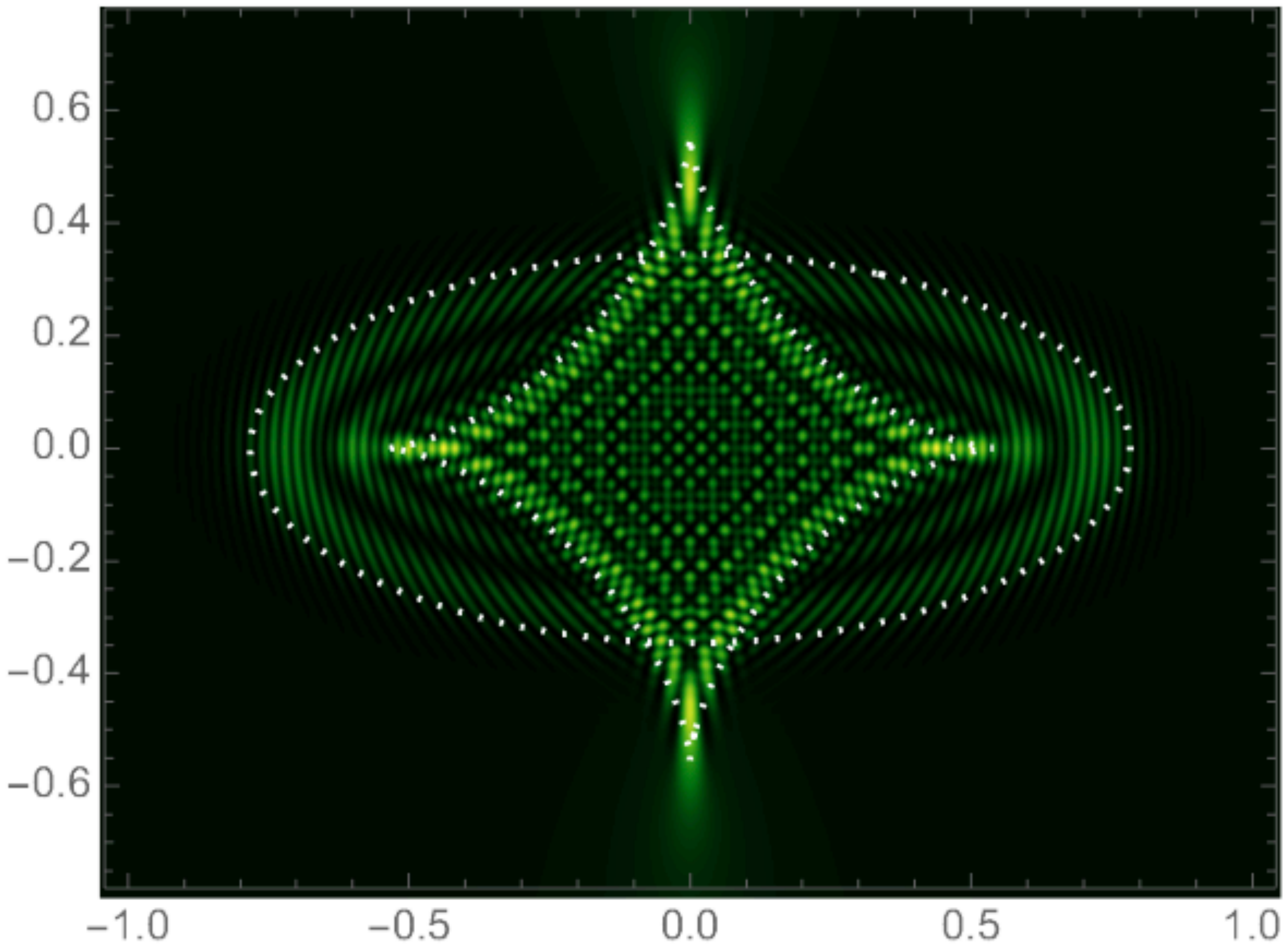
Multi-plane lensing

Radiation might encounter multiple lenses

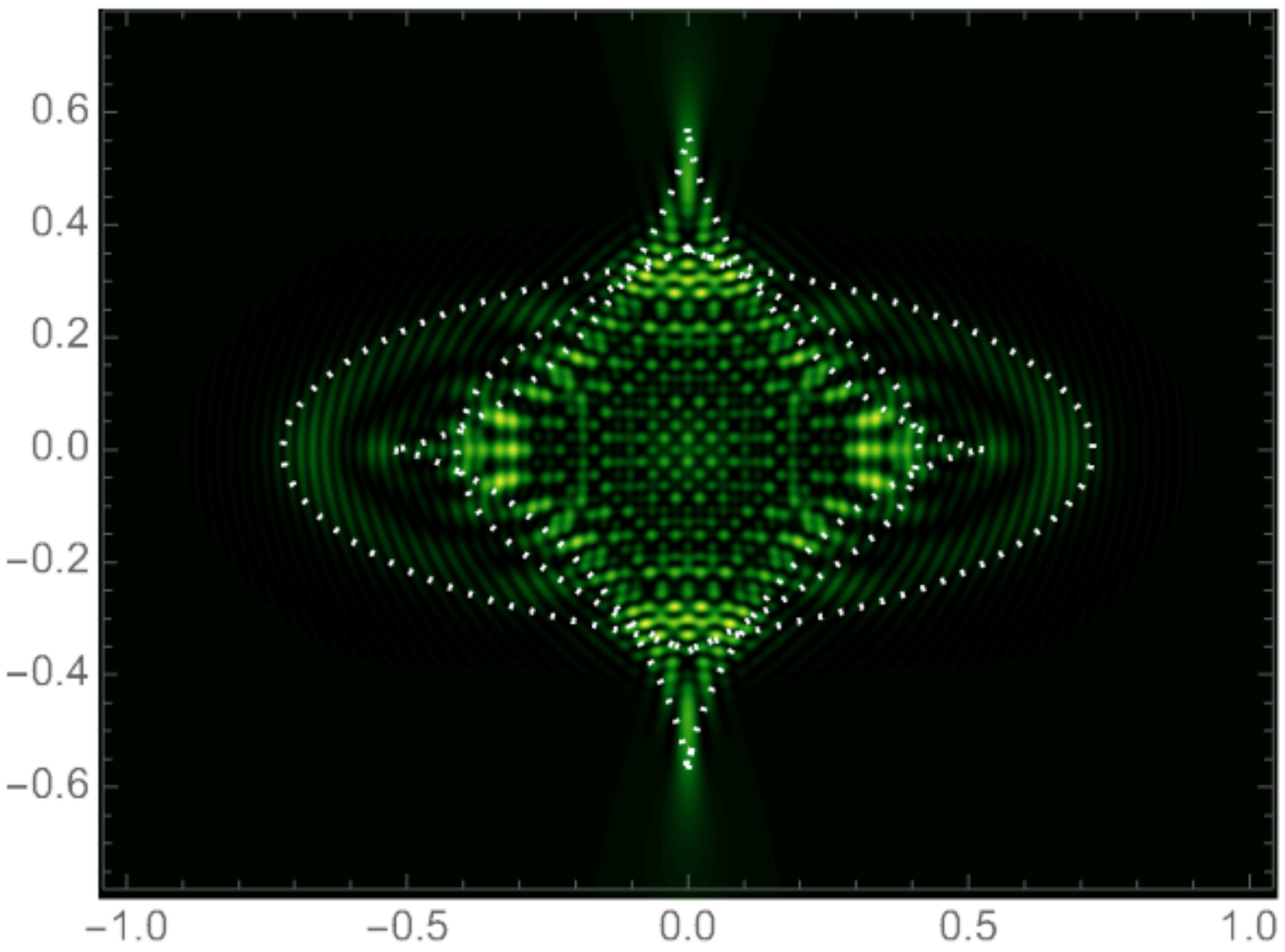


Multi-plane lensing

Radiation might encounter multiple lenses



(b) $I(\boldsymbol{x}_3) = |\int_{\mathbb{R}^d} e^{i\omega(T_1+T_2)} \mathrm{d}\boldsymbol{x}_2|^2$



(c) $I(\boldsymbol{x}_3) = |\Psi_3(\boldsymbol{x}_3)|^2$

Multi-plane lensing

Radiation might encounter multiple lenses

(a) $\nu = 50$ (b) $\nu = 100$ (c) $\nu = 200$

The no-boundary proposal

The Beginning of the World from the Point of View of Quantum Theory.

SIR ARTHUR EDDINGTON[1] states that, philosophically, the notion of a beginning of the present order of Nature is repugnant to him. I would rather be inclined to think that the present state of quantum theory suggests a beginning of the world very different from the present order of Nature. Thermodynamical principles from the point of view of quantum theory may be stated as follows: (1) Energy of constant total amount is distributed in discrete quanta. (2) The number of distinct quanta is ever increasing. If we go back in the course of time we must find fewer and fewer quanta, until we find all the energy of the universe packed in a few or even in a unique quantum.

Now, in atomic processes, the notions of space and time are no more than statistical notions; they fade out when applied to individual phenomena involving but a small number of quanta. If the world has begun with a single quantum, the notions of space and time would altogether fail to have any meaning at the beginning; they would only begin to have a sensible meaning when the original quantum had been divided into a sufficient number of quanta. If this suggestion is correct, the beginning of the world happened a little before the beginning of space and time. I think that such a beginning of the world is far enough from the present order of Nature to be not at all repugnant.

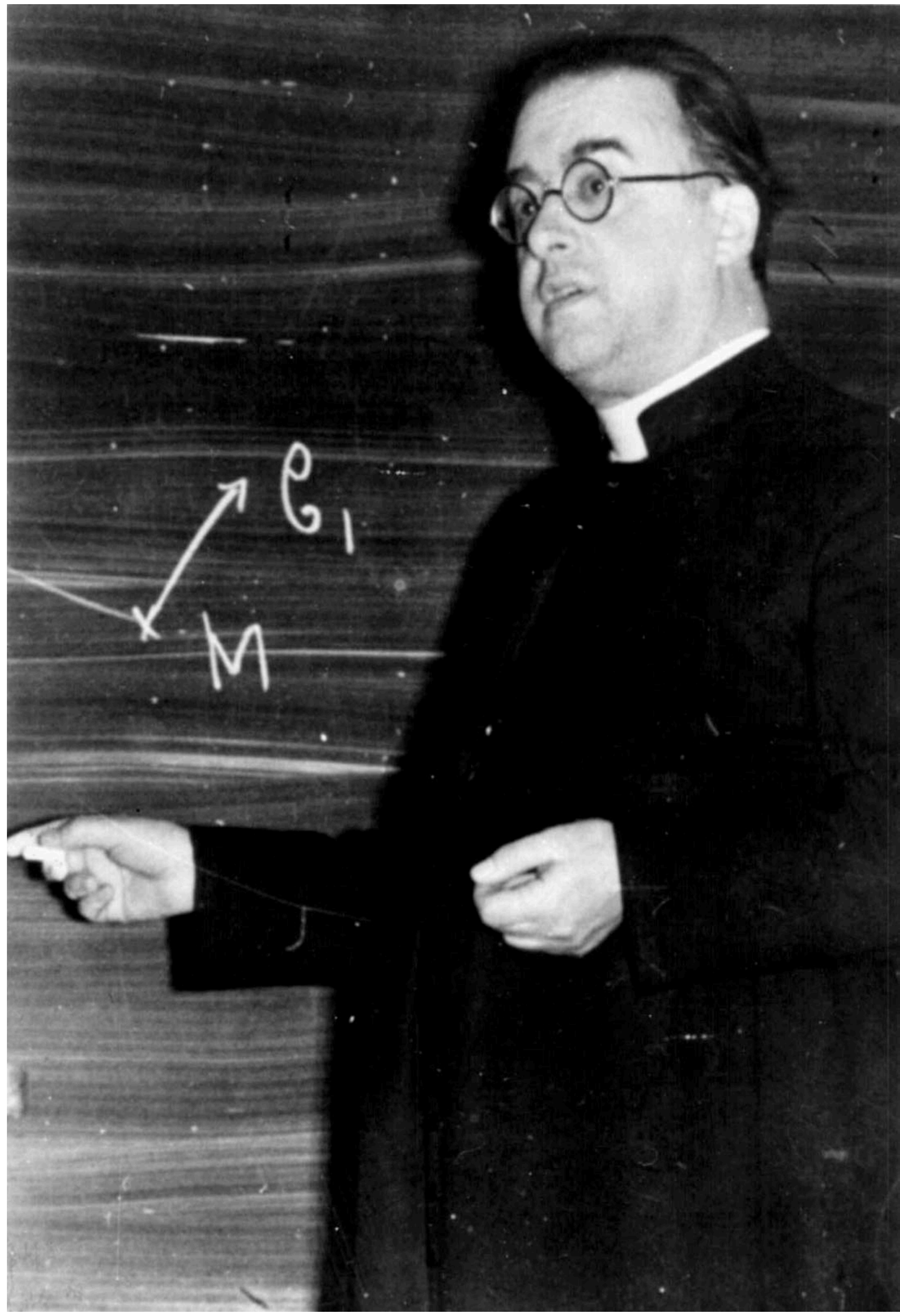
It may be difficult to follow up the idea in detail as we are not yet able to count the quantum packets in every case. For example, it may be that an atomic nucleus must be counted as a unique quantum, the atomic number acting as a kind of quantum number. If the future development of quantum theory happens to turn in that direction, we could conceive the beginning of the universe in the form of a unique atom, the atomic weight of which is the total mass of the universe. This highly unstable atom would divide in smaller and smaller atoms by a kind of super-radioactive process. Some remnant of this process might, according to Sir James Jeans's idea, foster the heat of the stars until our low atomic number atoms allowed life to be possible.

Clearly the initial quantum could not conceal in itself the whole course of evolution; but, according to the principle of indeterminacy, that is not necessary. Our world is now understood to be a world where something really happens; the whole story of the world need not have been written down in the first quantum like a song on the disc of a phonograph. The whole matter of the world must have been present at the beginning, but the story it has to tell may be written step by step.

G. LEMAÎTRE.

40 rue de Namur,
Louvain.

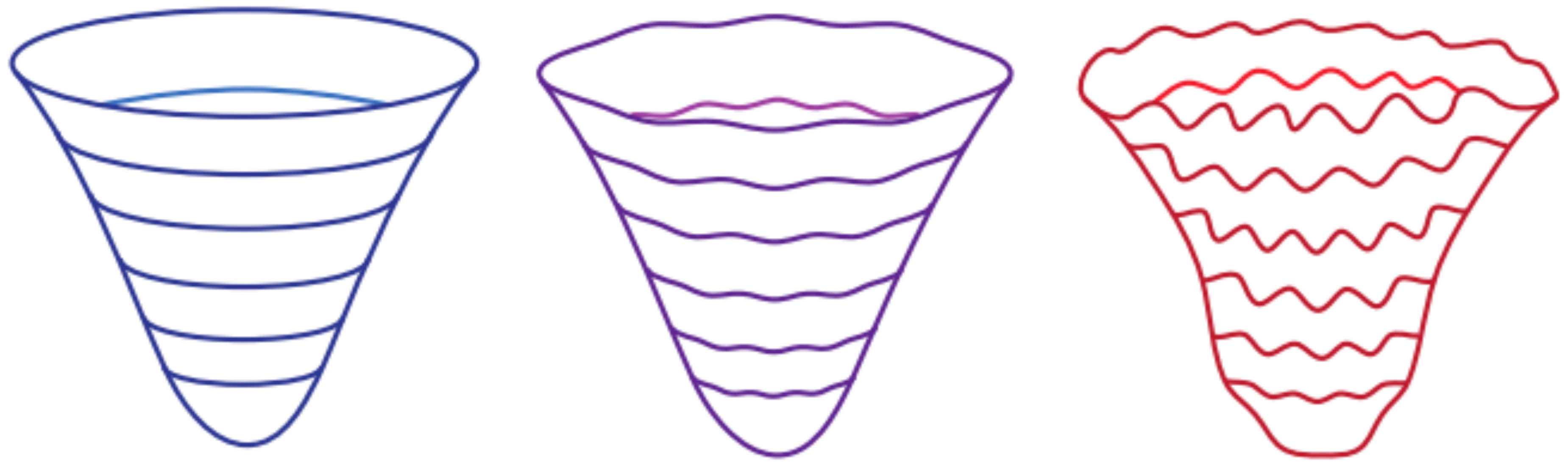
[1] NATURE, Mar. 21, p. 447.



Georges Lemaître, Nature **127,** 706 (1931)

The no-boundary proposal

- In the early 1980's, both Hartle, Hawking and Vilenkin developed famous models for the quantum big bang, known as the no-boundary and the tunnelling proposal.
- Nucleation of a classical closed Lambda-dominated universe out of a forbidden 'Euclidean' quantum phase.
- Aim: use the path integral for gravity to construct a predictive model for the initial conditions of our universe.
- In recent work, we used Picard-Lefschetz theory to study these proposals in the Lorentzian formulation.



Lorentzian Quantum Cosmology

No smooth beginning for spacetime

No Rescue for the No Boundary Proposal: Pointers to the Future of Quantum Cosmology

Inconsistencies of the New No-Boundary Proposal

Quantum Incompleteness of Inflation

No-boundary proposal

Minisuperspace $\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\Omega_3^2$

$$G[a_1; a_0] = \int_0^\infty \mathrm{d}N \int_{a_0}^{a_1} \mathcal{D}a \, e^{iS[N,a]}$$

with the Einstein-Hilbert action for a Λ-dominated universe

$$S = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\,(R - 2\Lambda) = 2\pi^2 \int_0^1 \mathrm{d}t\, N \left[-3a\frac{\dot{a}^2}{N^2} + 3ka - a^3\Lambda\right]$$

Redefining the lapse $N \mapsto N/a$, the action is quadratic in $q = a^2$

$$S = 2\pi^2 \int_0^1 \mathrm{d}t \left[-\frac{3}{4N}\dot{q}^2 + N(3k - \Lambda q)\right]$$

The propagator $G[q_1; q_0] \propto \int_0^\infty \frac{\mathrm{d}N}{\sqrt{N}} e^{iS_0[q_1;q_0;N]}$ with the classical action

$$S_0[q_1; q_0; N] = N^3\frac{\Lambda^2}{36} + N\left[-\frac{\Lambda}{2}(q_0 + q_1) + 3k\right] + \frac{1}{N}\left[-\frac{3}{4}(q_1 - q_0)^2\right]$$

Halliwell & Louko (1989), Brown & Martinez (1990)

No-boundary proposal

The propagator is an oscillatory integral

$$G[q_1; q_0 = 0] \propto \int_0^\infty \frac{\mathrm{d}N}{\sqrt{N}} e^{iS_0[q_1;q_0=0;N]}$$

Hartle and Hawking: 'The oscillatory integral [for the Lorentzian propagator] is not well defined but can be made so by rotating the time to imaginary values.'

Euclidean instead of Lorentzian gravity: rotate $N \to \pm iN$ and integrate

$$G_E[q_1; q_0 = 0] \propto \int_0^\infty \frac{\mathrm{d}N}{\sqrt{N}} e^{\pm S_0[q_1;q_0=0;N]}$$

No-boundary proposal

The Euclidean propagator is divergent due to the conformal factor problem. Physical considerations lead Hartle and Hawking to the propagator

$$G[q_1; q_0 = 0] \propto \int_{-\infty}^{\infty} \frac{\mathrm{d}N}{\sqrt{N}} e^{\pm S_0[q_1; q_0=0; N]}$$

which does satisfy the Wheeler-DeWitt equation $\hat{H}G = 0$. The integral was approximated with the saddle point approximation

$$|G[q_1; q_0 = 0]|^2 \propto e^{+\frac{12\pi^2}{\Lambda}} \sim e^{+\frac{12\pi^2}{V(\varphi_1)}}$$

When including a scalar field (gravitons)

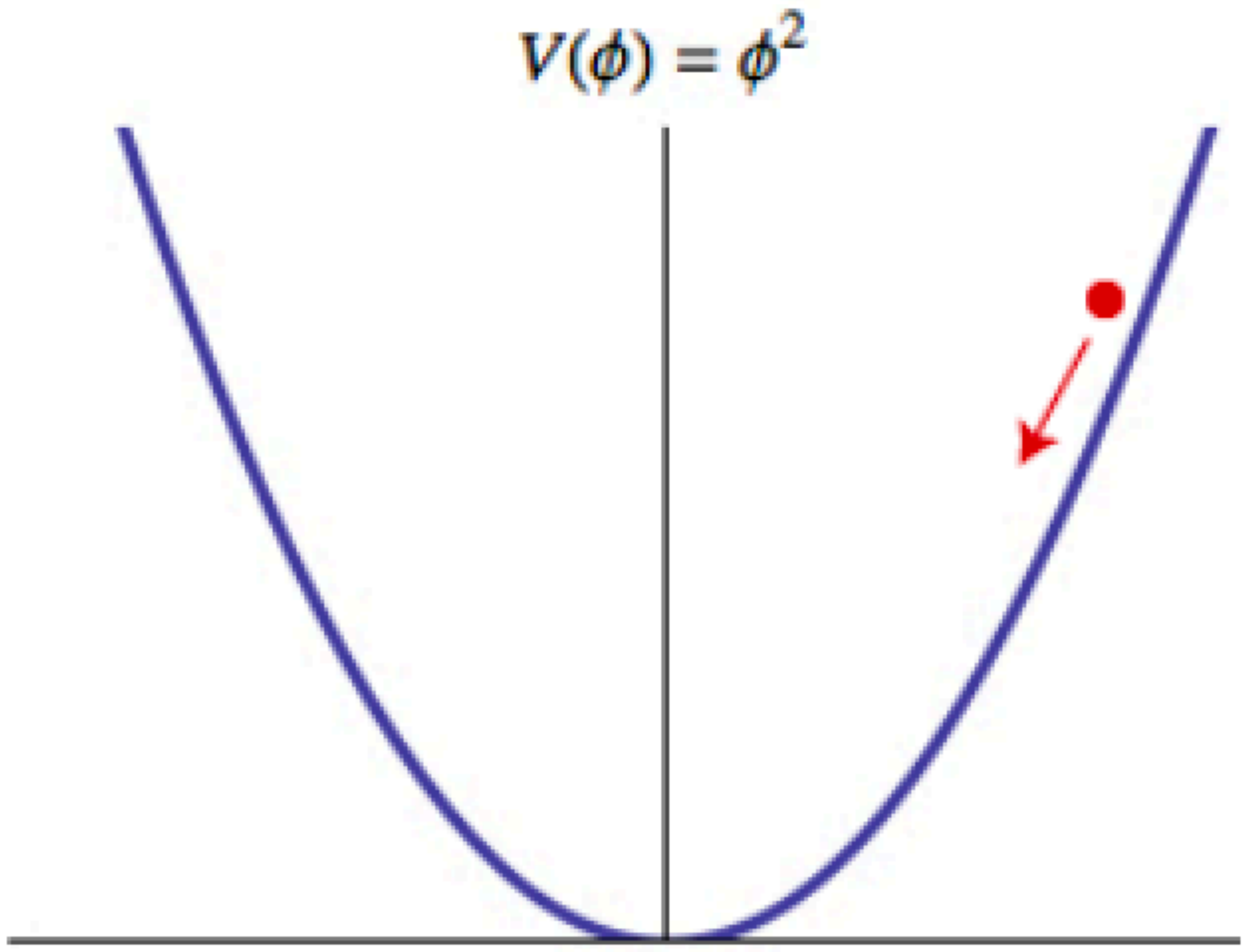
$$|G[q_1, \phi_1; q_0 = 0]|^2 \propto e^{+\frac{12\pi^2}{\Lambda} - c\phi_1^2} \sim e^{+\frac{12\pi^2}{V(\varphi)} - c\phi_1^2}$$

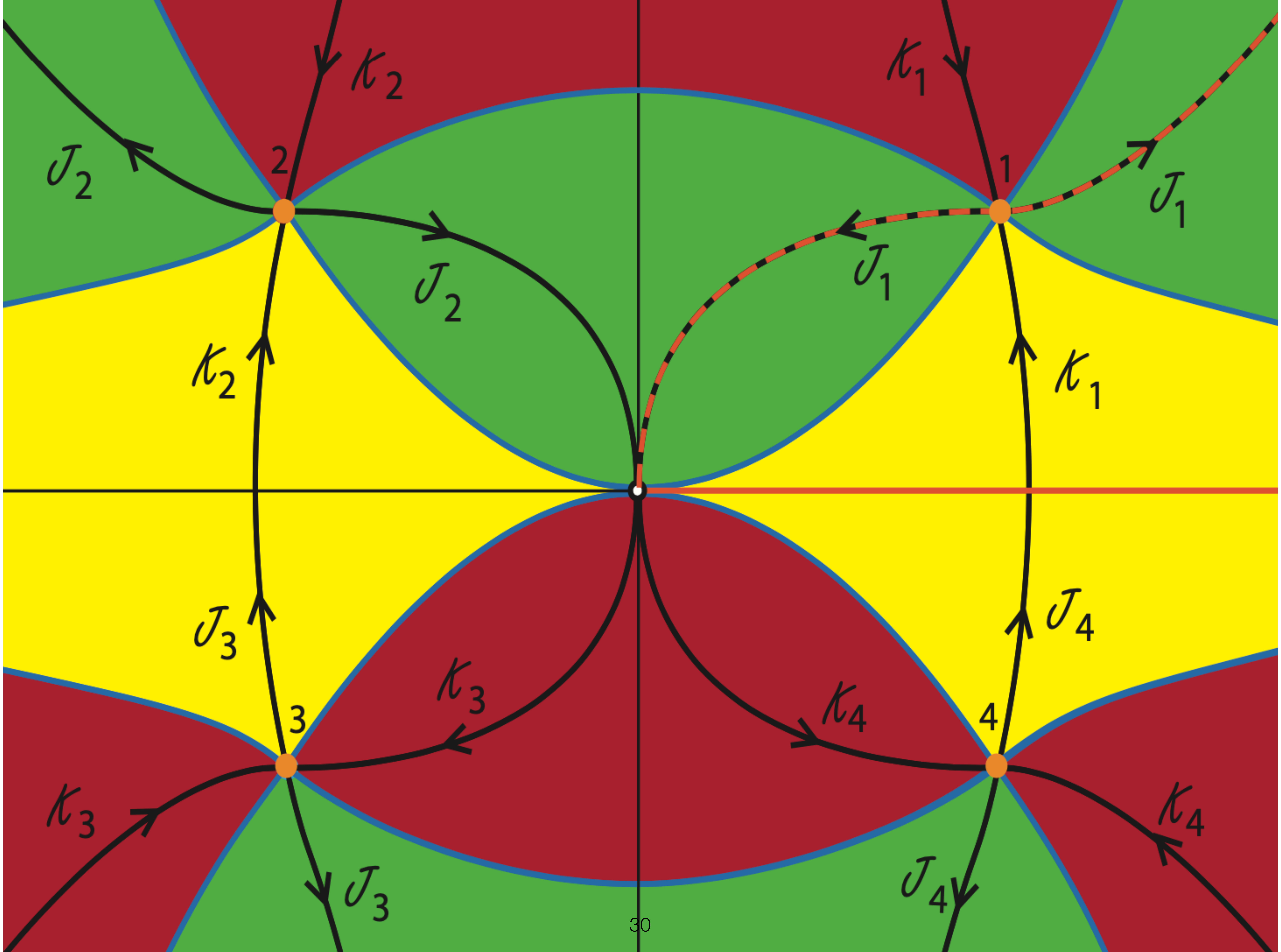
No-boundary proposal

The Hartle-Hawking initial wave function $|G[q_1, \phi_1]|^2 \propto e^{+\frac{12\pi^2}{V(\varphi)} - c\phi_1^2}$

- Gaussian fluctuations
- Bottom of instead of the top of the the potential

Resolved with anthropic principle




$\mathcal{K}_2$
$\mathcal{K}_1$
2
1
$\mathcal{J}_2$
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$\mathcal{K}_4$
3
4
$\mathcal{K}_3$
$\mathcal{K}_4$
$\mathcal{J}_3$
$\mathcal{J}_4$

No-boundary proposal

Weighting *no-boundary proposal*:

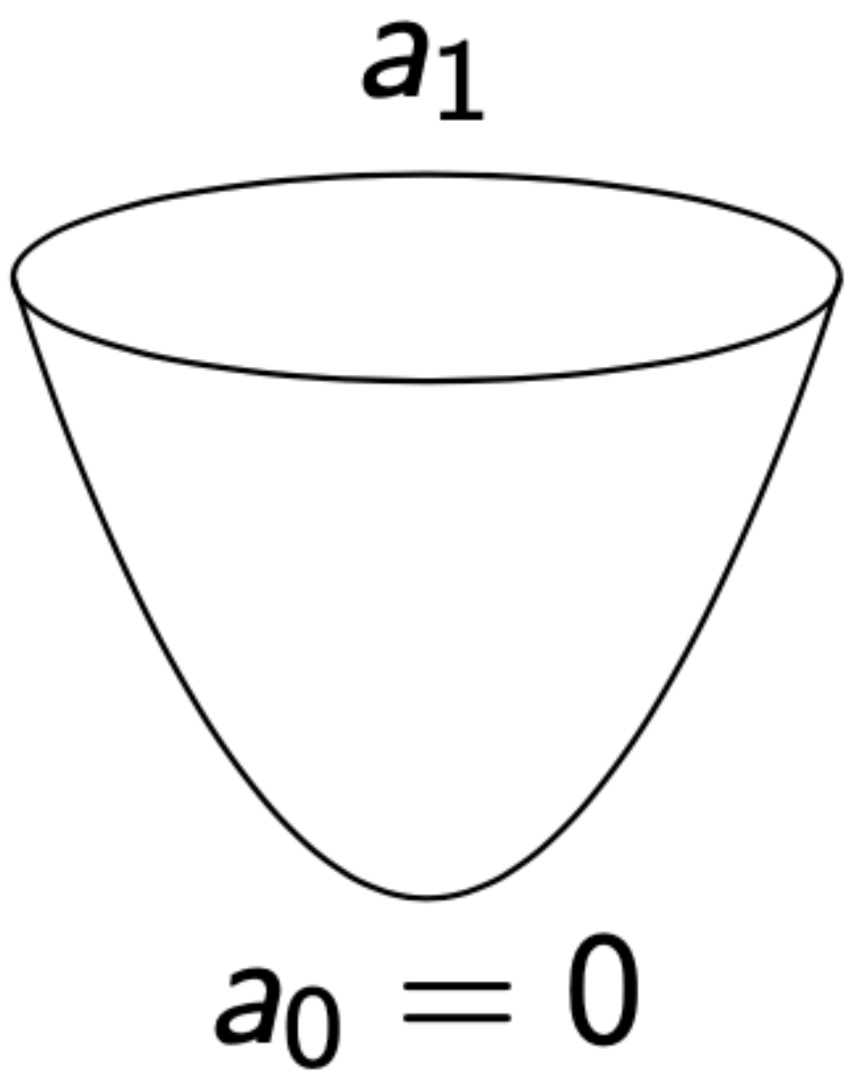
- Hartle-Hawking result

$$
\begin{aligned}
|G[a_1;0]|^2 &\propto e^{+12\pi^2/(\hbar\Lambda)} \\
&\propto e^{+12\pi^2/(\hbar V(\varphi))}
\end{aligned}
$$

- Picard-Lefschetz theory result

$$
\begin{aligned}
|G[a_1;0]|^2 &\propto e^{-12\pi^2/(\hbar\Lambda)} \\
&\propto e^{-12\pi^2/(\hbar V(\varphi))}
\end{aligned}
$$

- Same result as Vilenkin's tunneling proposal
- Lorentzian gravity $\neq$ Euclidean gravity



The no-boundary proposal

The propagator

$$G[q_1, \phi_1; q_0, \phi_0] = \int_{0^+}^{\infty} \mathrm{d}N \int_{q_0}^{q_1} \mathcal{D}q \int_{\phi_0}^{\phi_1} \mathcal{D}\phi \, e^{i(S^{(0)}[q,N]+S^{(2)}[q,\phi,N])/\hbar}$$

The gravitational wave action

$$S^{(2)} = \frac{1}{2}\int_0^1 N_s \mathrm{d}t \, \mathrm{d}^3x \left[q^2 \left(\frac{\dot{\phi}}{N_s} \right)^2 - l(l+2)\phi^2 \right]$$

with N_s the saddle point in the background.

The no-boundary proposal

Equation of motion

$$\ddot{\phi} + 2\frac{\dot{q}}{q}\dot{\phi} + \frac{N_s^2}{q^2} l(l+2)\phi = 0$$

is solved by $\phi(t) = \phi_1 F(t)/F(1)$ with

$$F(t) = \left(1 + \frac{i}{H^2 N_s t - i}\right)^{\frac{l}{2}} \left(1 - \frac{i}{H^2 N_s t - i}\right)^{-\frac{l+2}{2}} \left(1 - \frac{i(l+1)}{H^2 N_s t - i}\right)$$

with boundary conditions $\phi(0) = 0, \phi(1) = \phi_1$. The classical action

$$S^{(2)}[q_1, \phi_1; 0, 0] = \frac{1}{2}\int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{q^2}{N_s}\phi\dot{\phi}\right] = \frac{q_1^2}{2N_s}\phi_1^2 \frac{\dot{F}(1)}{F(1)}$$

$$= \frac{\phi_1^2}{2}\left[-i\frac{l(l+2)}{H}\sqrt{q_1} + \frac{l(l+1)(l+2))}{H^2} + \mathcal{O}\left(\frac{1}{\sqrt{q_1}}\right)\right]$$

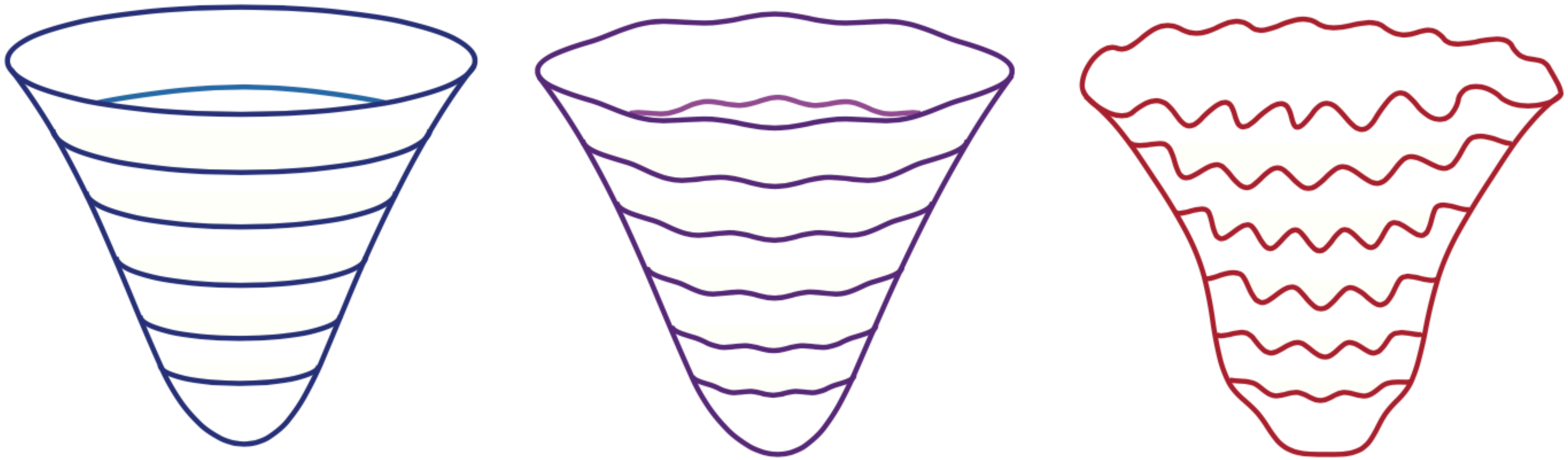
The no-boundary proposal

The total propagator factorizes $G[q_1, \phi_1; 0, 0] = G[q_1; 0] G_\phi[\phi_1; 0]$, with the GW propagator

$$G_\phi[\phi_1; 0] \propto e^{\frac{l(l+1)(l+2)}{2\hbar H^2}\phi_1^2} \times phase$$

This is an inverse Gaussian distribution

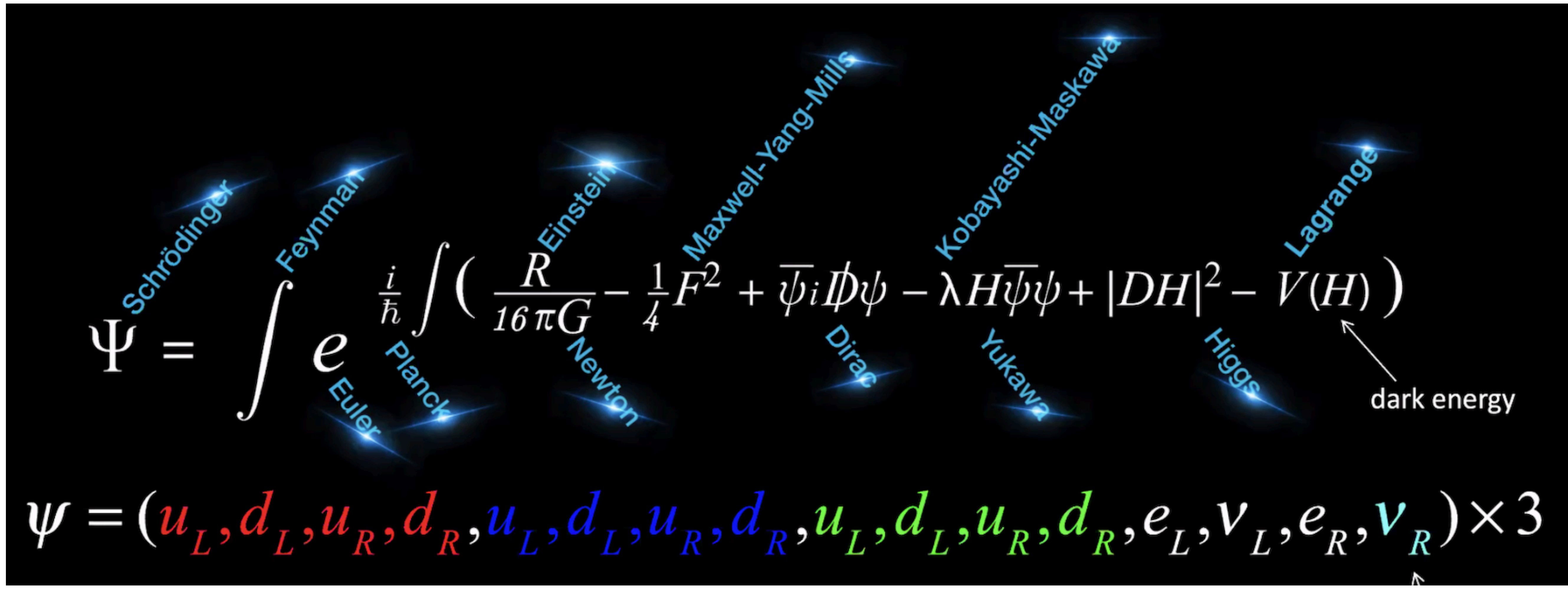
$$\left|G_\phi[\phi_1; 0]\right|^2 \propto e^{\frac{l(l+1)(l+2)}{\hbar H^2}\phi_1^2}$$



Lorentzian path integral

Is the path integral a true integral? Can we construct a rigorous definition?

- Feynman-Kac formula



$$K[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{i\int_0^T [\dot{x}^2 - V(x)]\mathrm{d}t/\hbar} \mathscr{D}x$$

Wick rotation: $t \mapsto -it$

$$G[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{-\int_0^T V(x)\mathrm{d}t/\hbar} \left[e^{-\int_0^T \dot{x}^2 \mathrm{d}t/\hbar} \mathscr{D}x \right]$$

$$\equiv \int_{x(0)=x_0}^{x(T)=x_1} e^{-\int_0^T V(x)\mathrm{d}t/\hbar} \, \mathrm{d}\mu_{x_1,x_0}^{1/\hbar}(x)$$

when the potential is bounded from below

$$V(x) \geq c$$

Lorentzian path integral

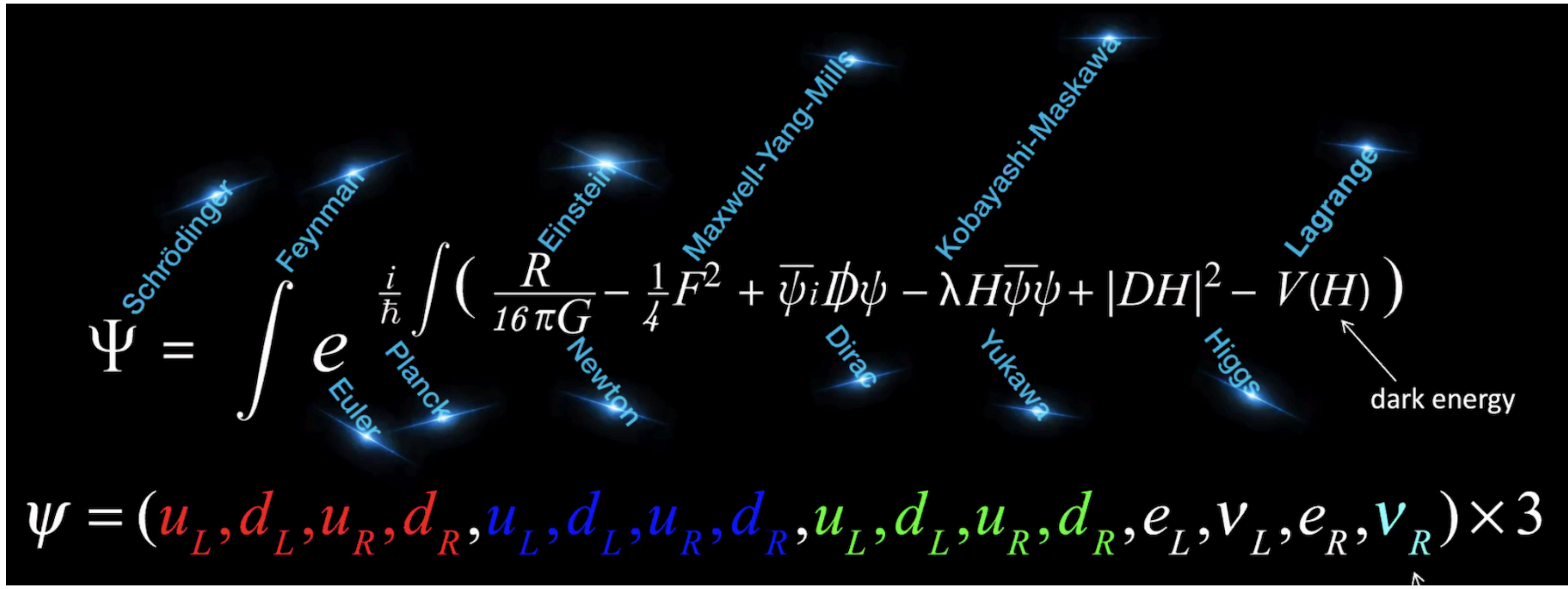
If we want to use complex analysis (instantons) to study path integrals, the \`path integral' better be an integral

- Picard-Lefschetz theory

$$K[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{i\int_0^T [\dot{x}^2 - V(x)]\mathrm{d}t/\hbar} \mathscr{D}x$$

$$\equiv \sum_i e^{iH(\bar{x}_i)/\hbar} \int_{\mathcal{J}_i} e^{h(\delta x_i/\hbar)}\, \mathrm{d}\mu_i(\delta x_i)$$

$x(t) = \bar{x}_i(t) + \delta x_i(t)$



- Which instantons are relevant?

- How does the h-function decrease along the thimble?

Quantum geometrodynamics

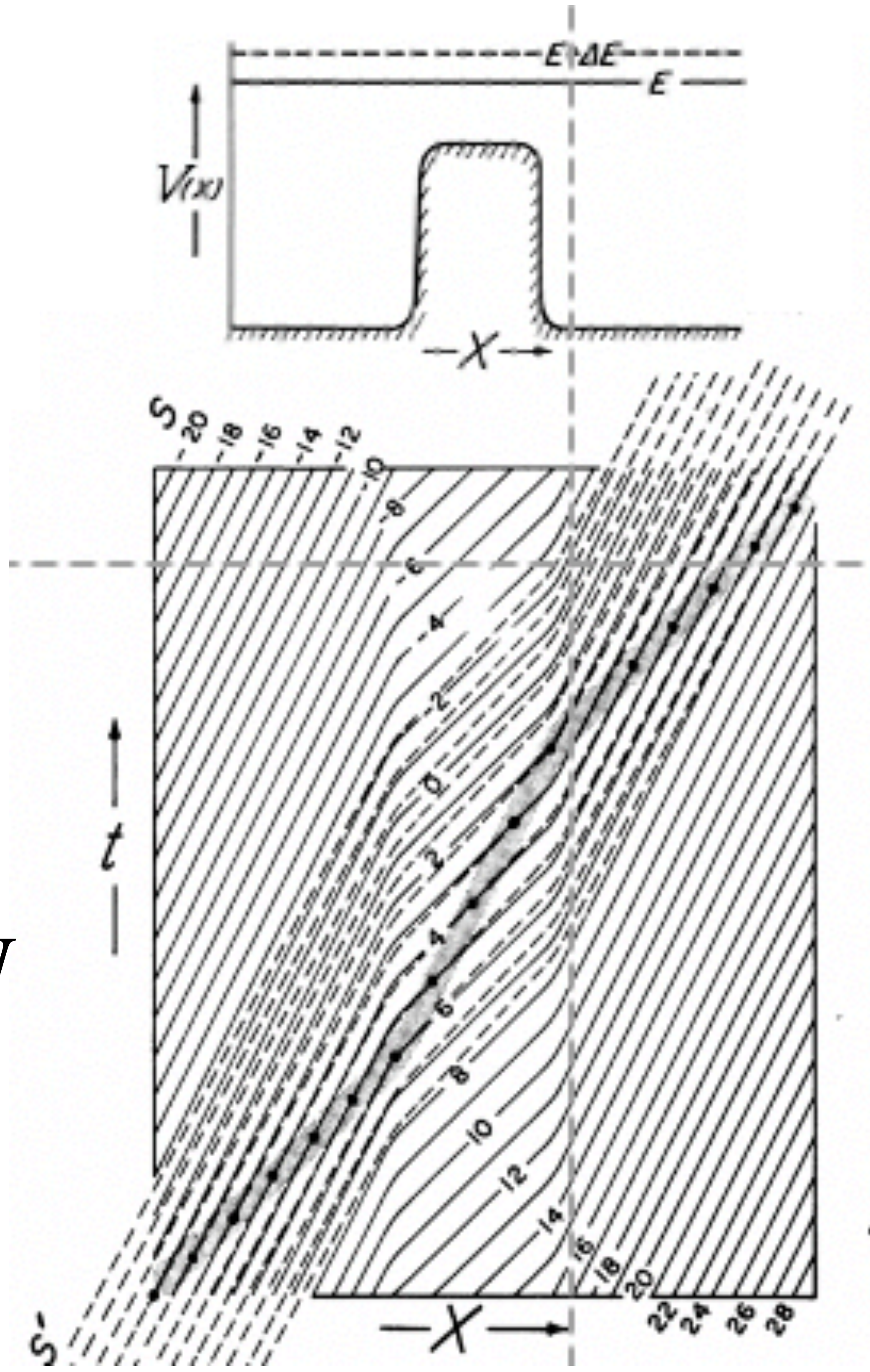
Wheeler: A classical trajectory emerges as an interference phenomena in quantum mechanics. Classical spacetime spacetime should emerge as an interference effect in superspace

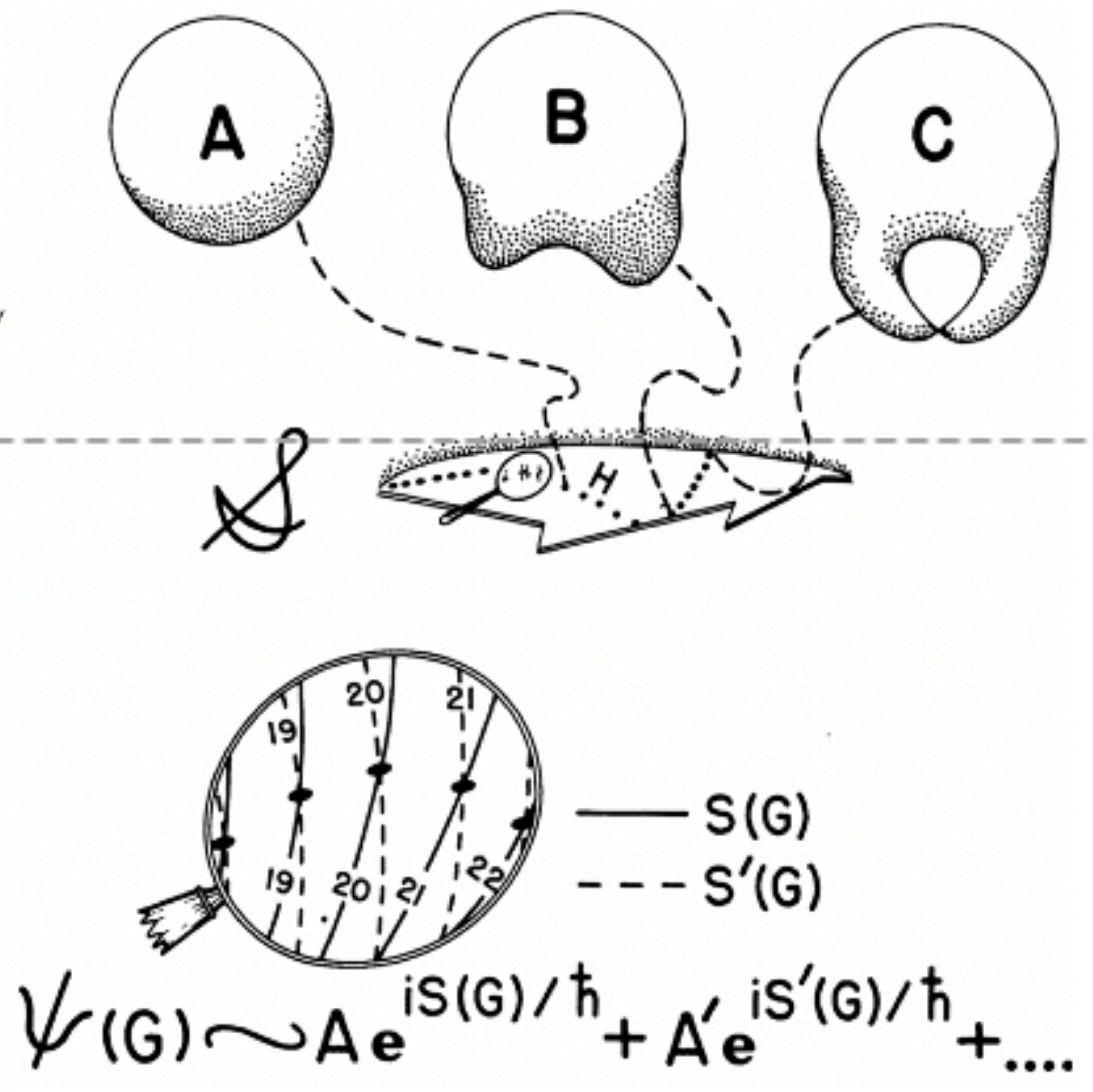
- The Wheeler-DeWitt equation

$$\hat{\mathscr{H}}_0\Psi[\mathscr{G}^{(3)}] = 0 \qquad \hat{\mathscr{H}}_i\Psi[\mathscr{G}^{(3)}] = 0$$

- The path integral over spacetimes

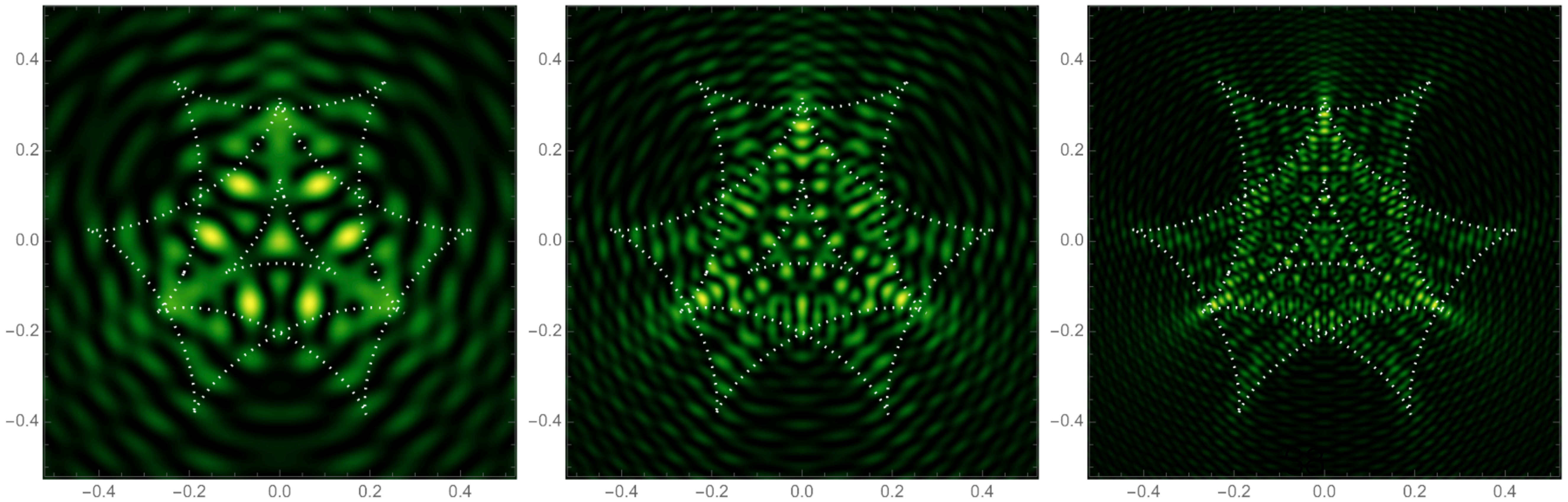
$$K[\mathscr{G}^{(3)}_1, \mathscr{G}^{(3)}_0] = \int_0^\infty \int_{\mathscr{G}^{(3)}_0}^{\mathscr{G}^{(3)}_1} e^{iS_{EH}[\mathscr{G};N]/\hbar}\, \mathscr{D}\mathscr{G}\,\mathrm{d}N$$

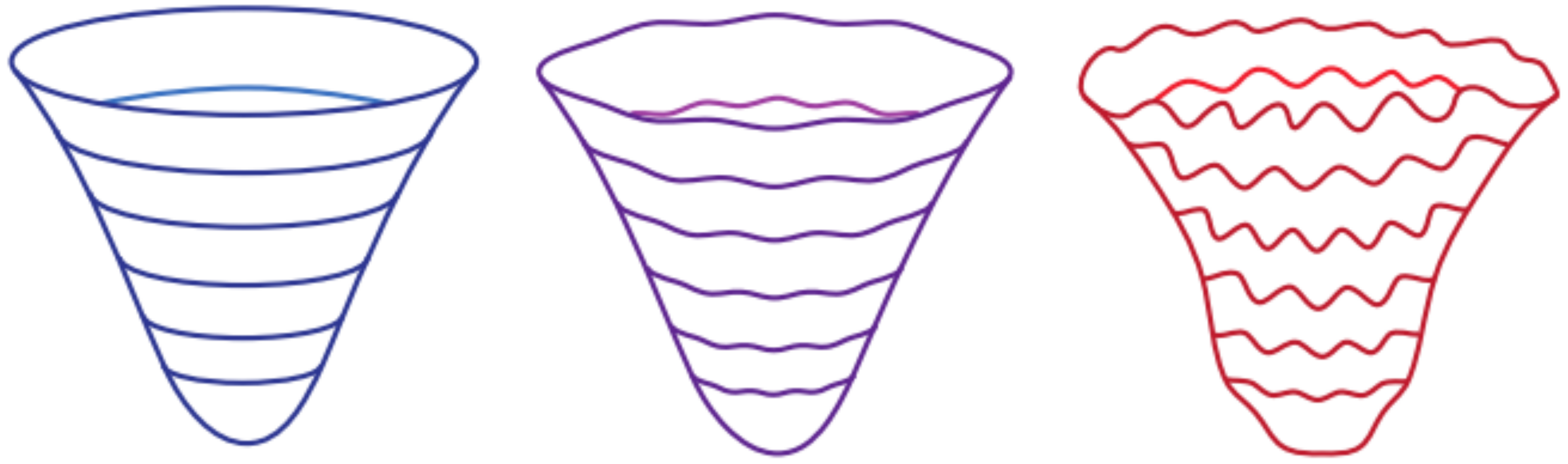




Summary

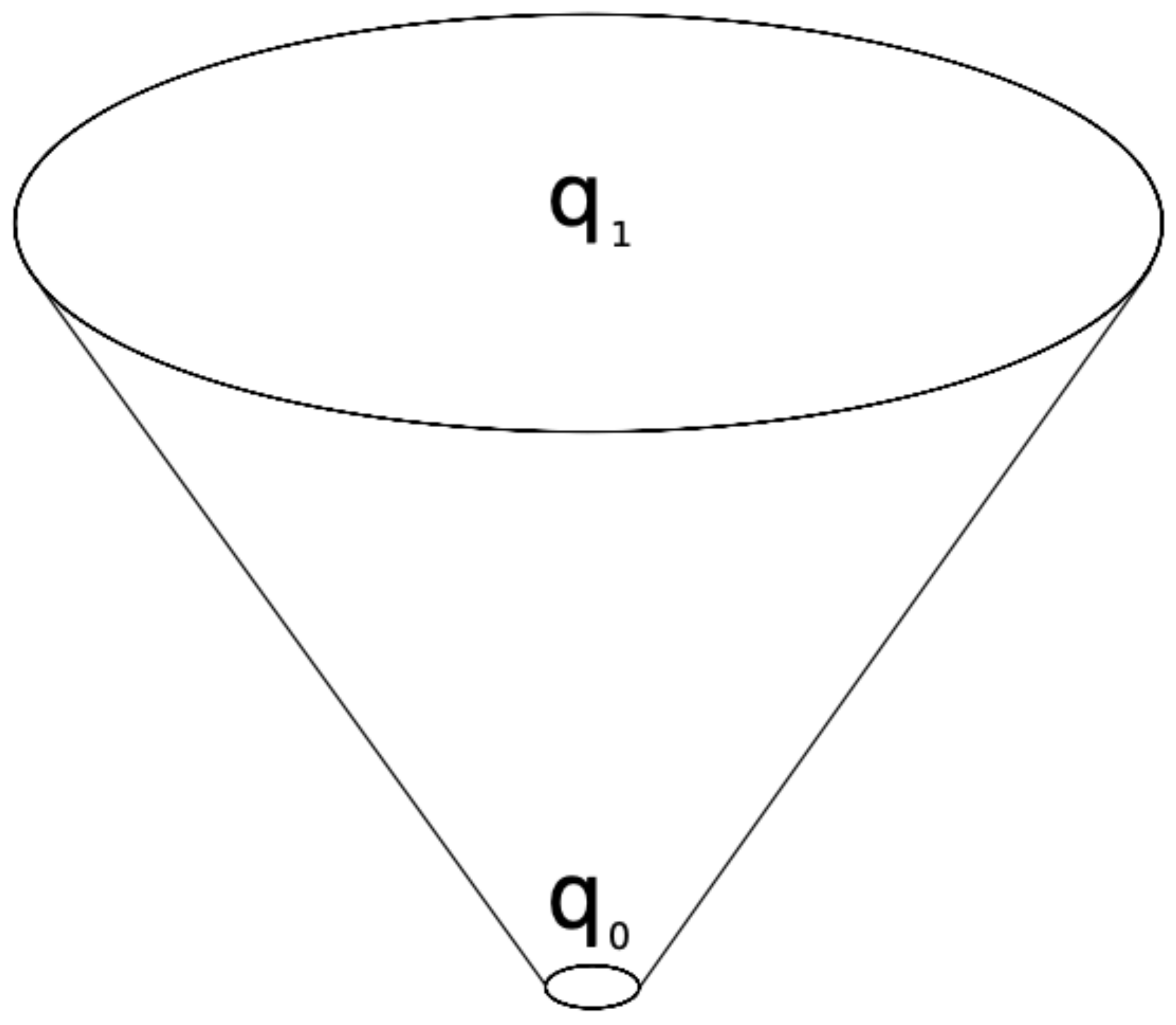
- Interference and caustics are among the most universal phenomena in physics
- Using Picard-Lefschetz theory we can construct a rigorous definition and efficiently evaluate oscillatory integrals
- With these techniques we can model lensing in wave optics and study Lorentzian quantum gravity



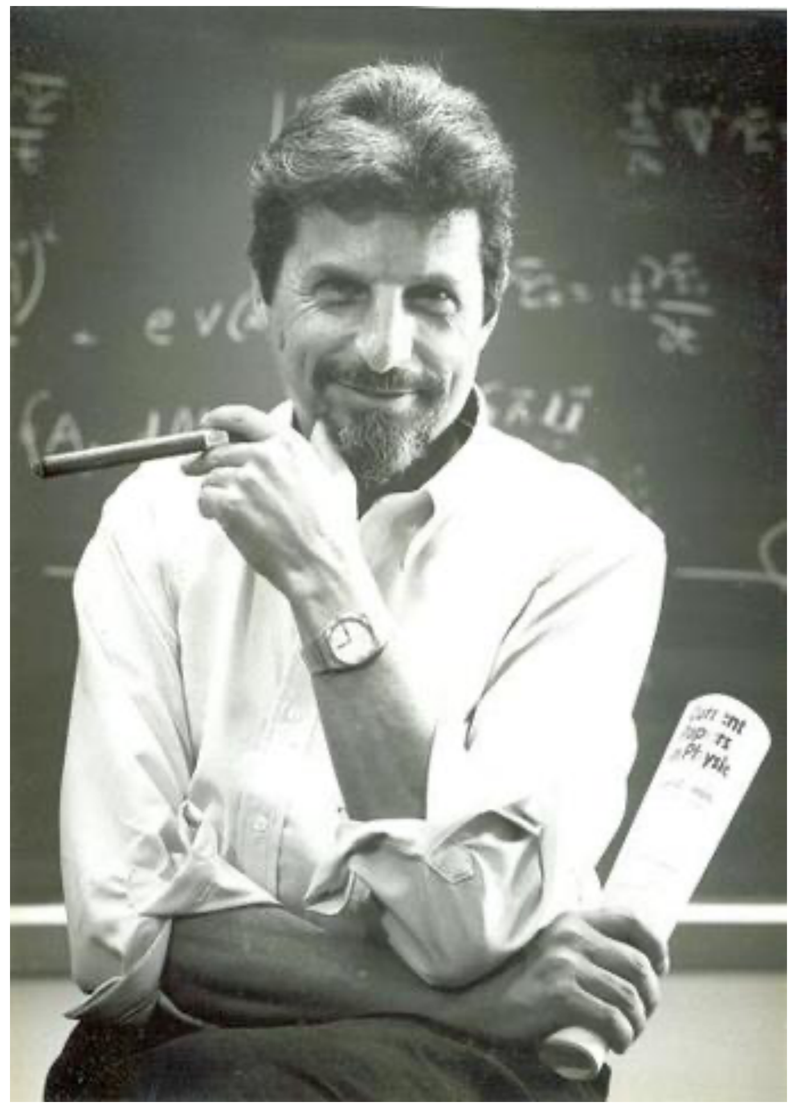


Bunch-Davies vacuum

- The Bunch-Davies vacuum is another popular initial state of our universe, normally defined in quantum field theory on classical spacetime
- It can also be defined from the path integral for gravity when considering a flat Lambda-dominated universe, in the limit of $q_0 \to 0$
- We showed that the Bunch-Davies proposal in quantum spacetime suffers from similar ambiguities



Relativistic weak value theory



Aharonov developed weak value theory for non-relativistic quantum mechanics. I extend it to relativistic systems:

- Couple the observable to a Von Neumann pointer to model the observer quantum mechanically
- Consider the limit weak coupling to remove interference on the experiment

$$S_p[X, P] = \int_0^T \mathrm{d}t \left[-X\dot{P} - \mathcal{H}_p\right] \qquad \mathcal{H}_p = \frac{P^2}{2M} + gPO[x^\mu]$$

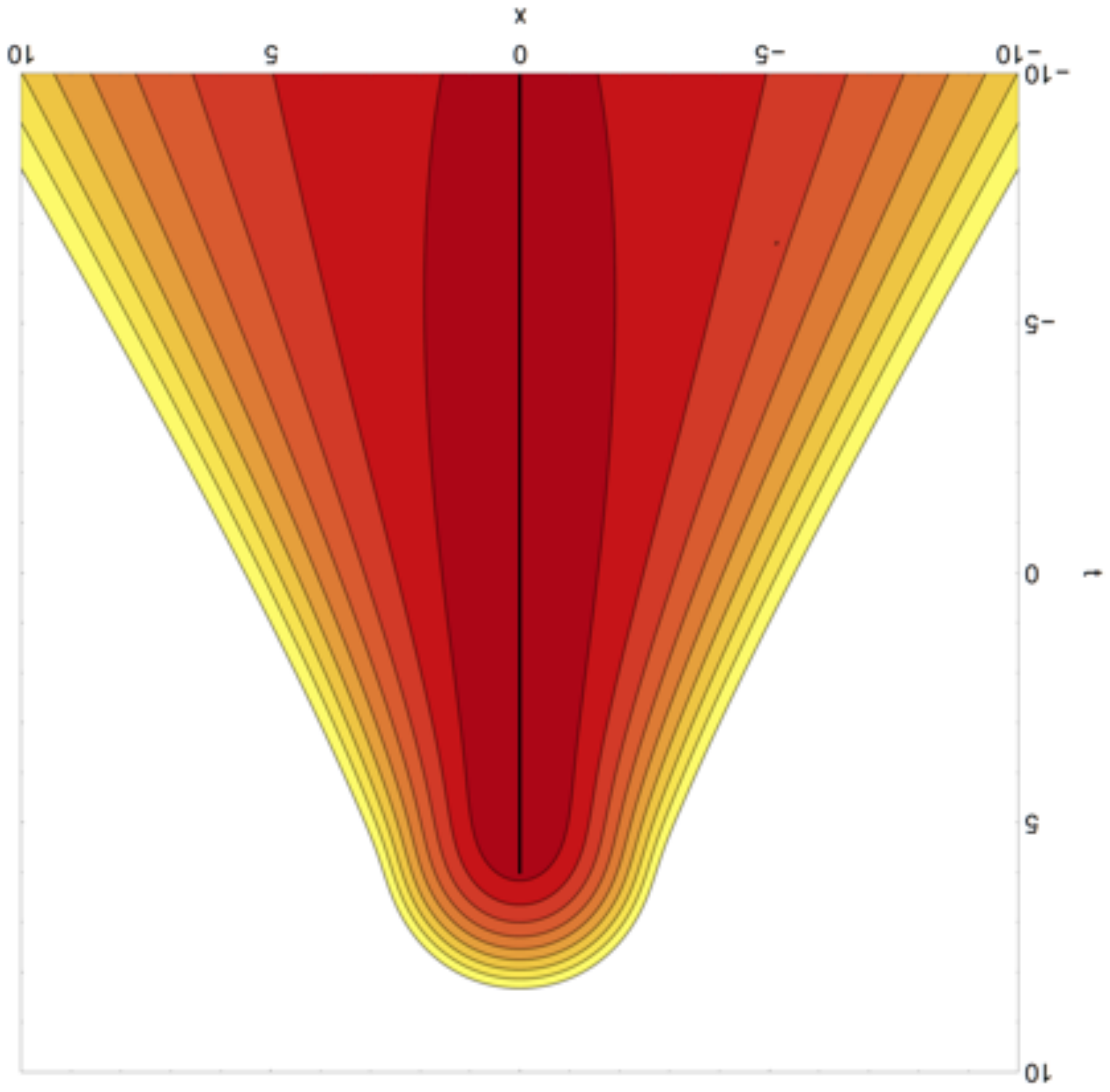
$$O_w = \frac{\int_{0^+}^{\infty} \mathrm{d}s \int_{\psi_0}^{\psi_1} \mathcal{D}x^\mu e^{iS/\hbar} O[x^\mu]}{\int_{0^+}^{\infty} \mathrm{d}s \int_{\psi_0}^{\psi_1} \mathcal{D}x^\mu e^{iS/\hbar}} \quad \langle X_1 \rangle = \langle X_0 \rangle + gT\ \mathrm{Re}[O_w]$$

- Consider the Klein-Gordon equation in an electric field

$$\hat{H}\phi = 0 \qquad \hat{H} = D_\mu D^\mu + m^2$$

$$G[x_1^\mu; x_0^\mu] = \int_{0^+}^{\infty} \mathrm{d}s \langle x_1^\mu | e^{-is\hat{H}/\hbar} | x_0^\mu \rangle$$

- Smoothed with an initial seed



$$\varphi(x_1^\mu) = \int_{0^+}^{\infty} \mathrm{d}s \langle x_1^\mu | e^{-is\hat{H}/\hbar} | \psi_0 \rangle$$

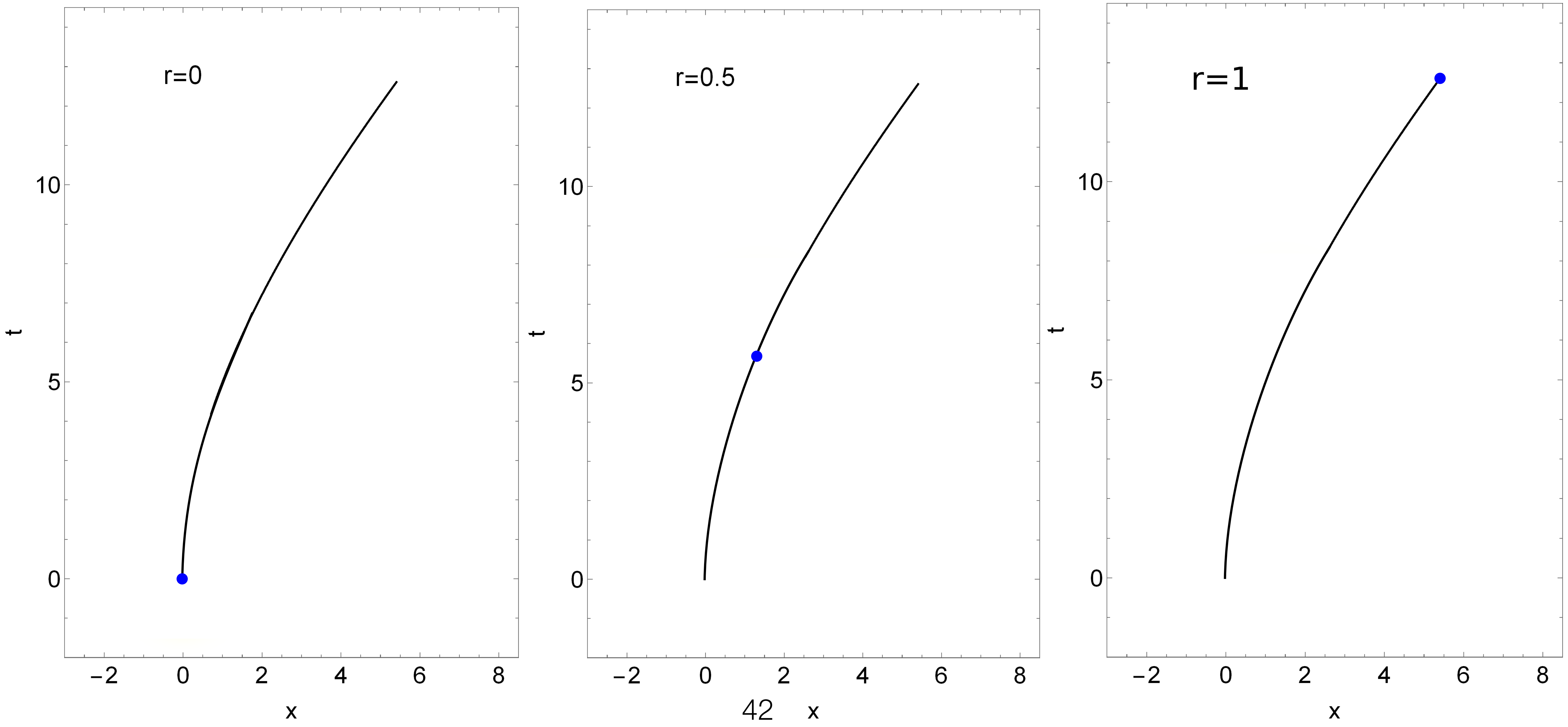
$$= \int \mathrm{d}x_0^\mu G[x_1^\mu; x_0^\mu] \psi_0(x_0^\mu)$$

satisfying the inhomogeneous equation

$$\hat{H}\varphi(x^\mu) = -i\hbar\psi_0(x^\mu)$$

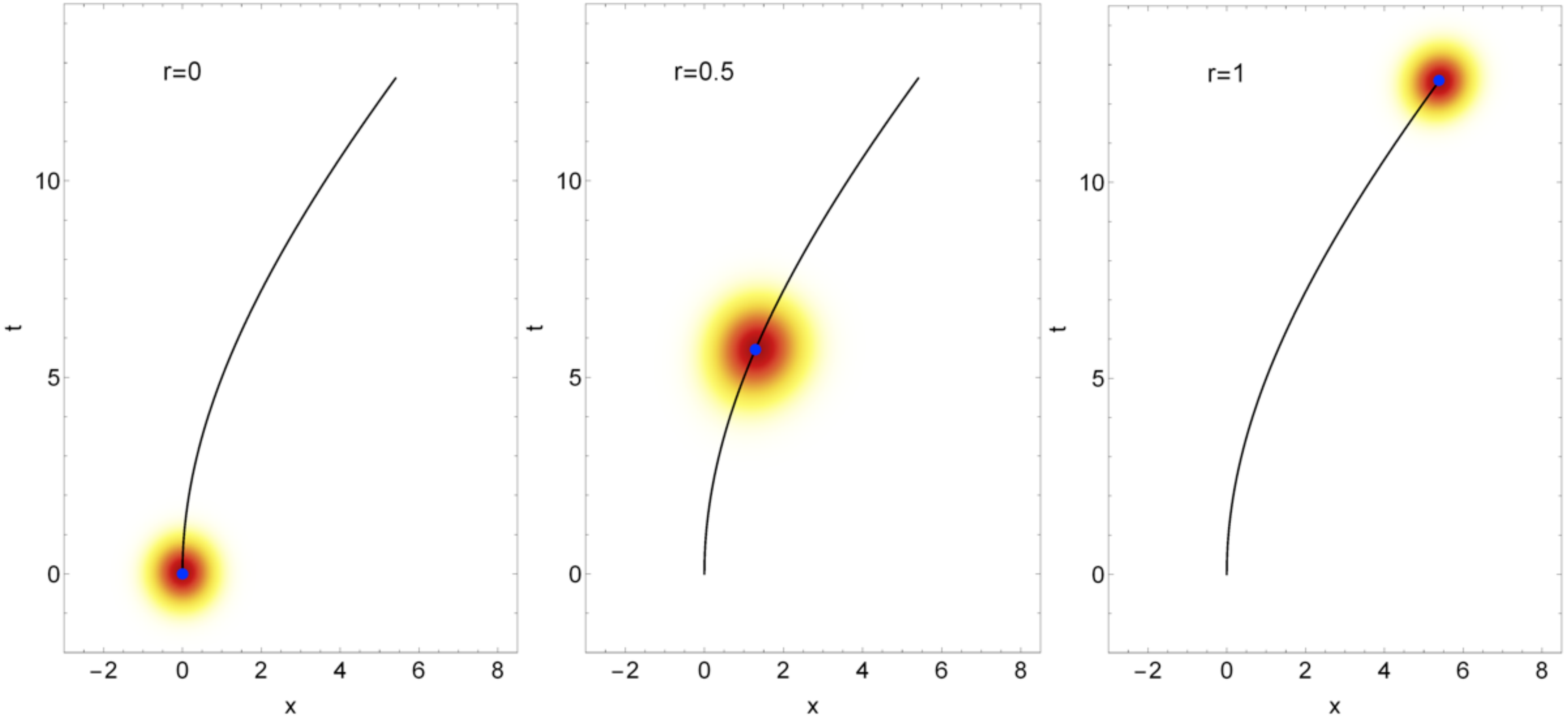
- When the observable is the position, we obtain the quantum generalization of (complex) classical trajectories tracing the most likely world-line

$$x_w(r) = \frac{\int_{0^+}^{\infty} \mathrm{d}s \int_{\psi_0}^{\psi_1} \mathcal{D}x^\mu e^{iS[x^\mu]/\hbar} x^\mu(r)}{\int_{0^+}^{\infty} \mathrm{d}s \int_{\psi_0}^{\psi_1} \mathcal{D}x^\mu e^{iS[x^\mu]/\hbar}}$$

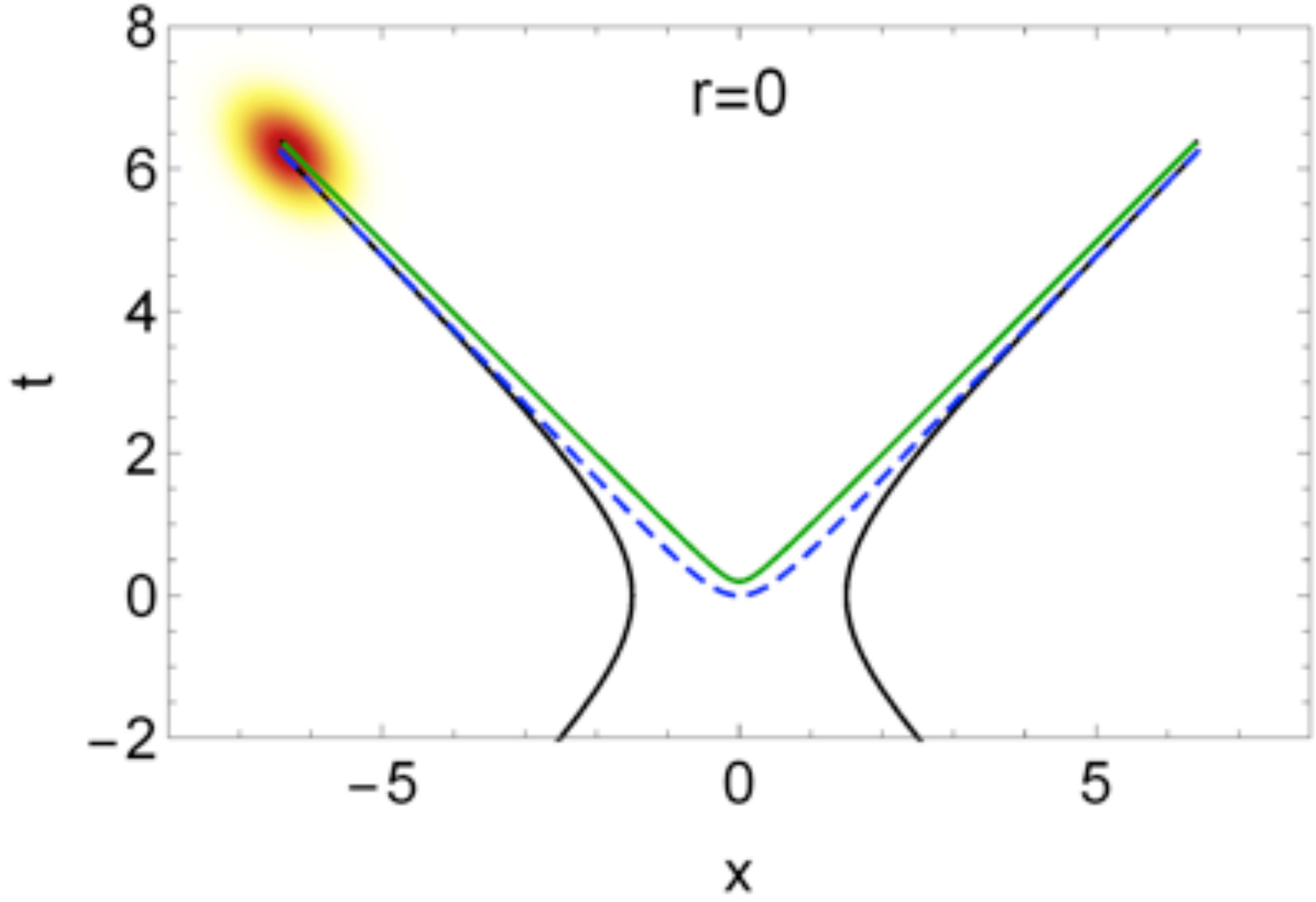


- The relativistic Aharonov-Bergmann-Lebowitz formula

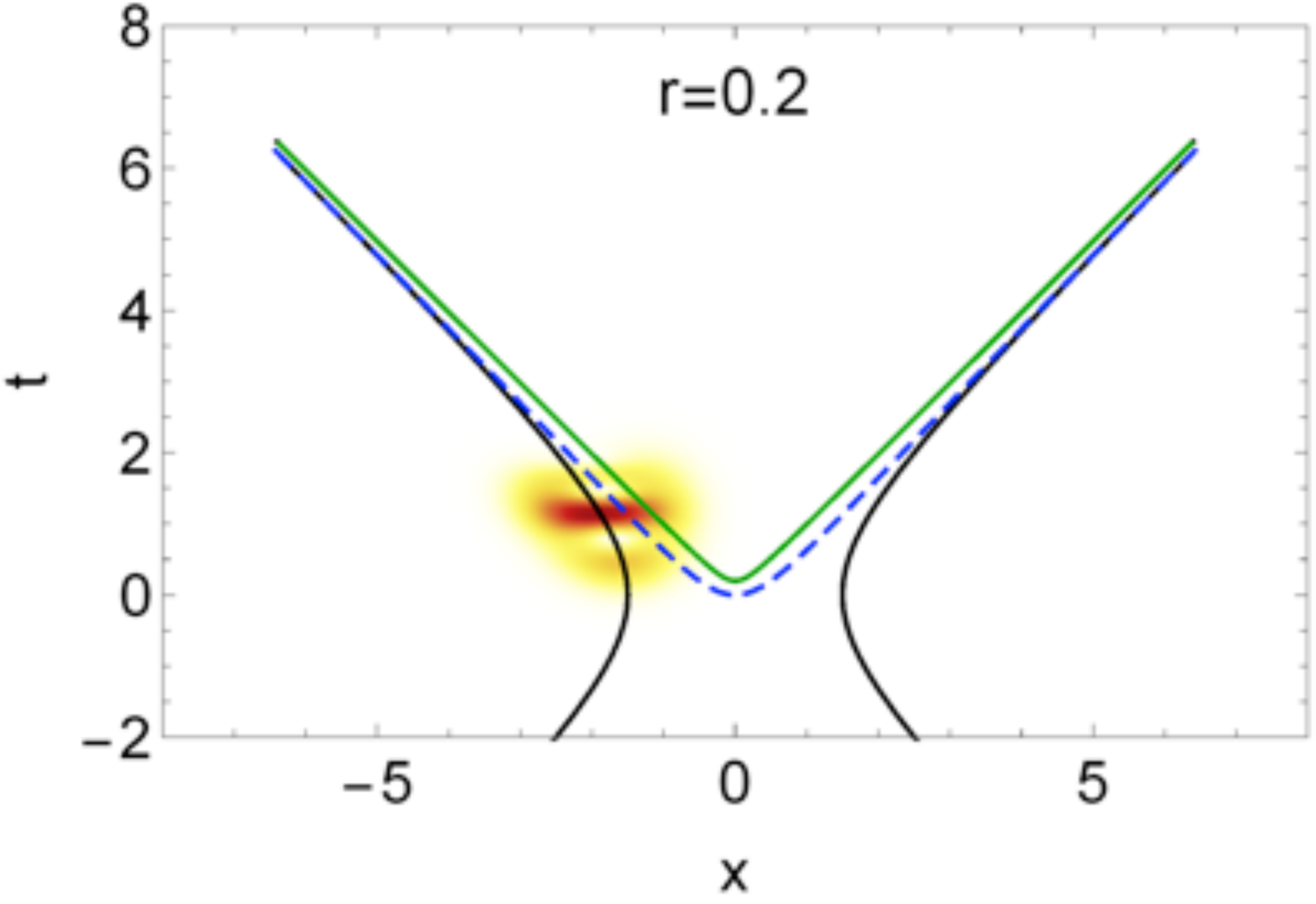
$$P_{ABL}[x^\mu_m; r] \propto \left| \int_{0^+}^{\infty} \mathrm{d}s \int_{\psi_0}^{\psi_1} \mathcal{D}x^\mu e^{iS[x^\mu]/\hbar} \delta(x^\mu_m - x^\mu(r)) \right|^2$$



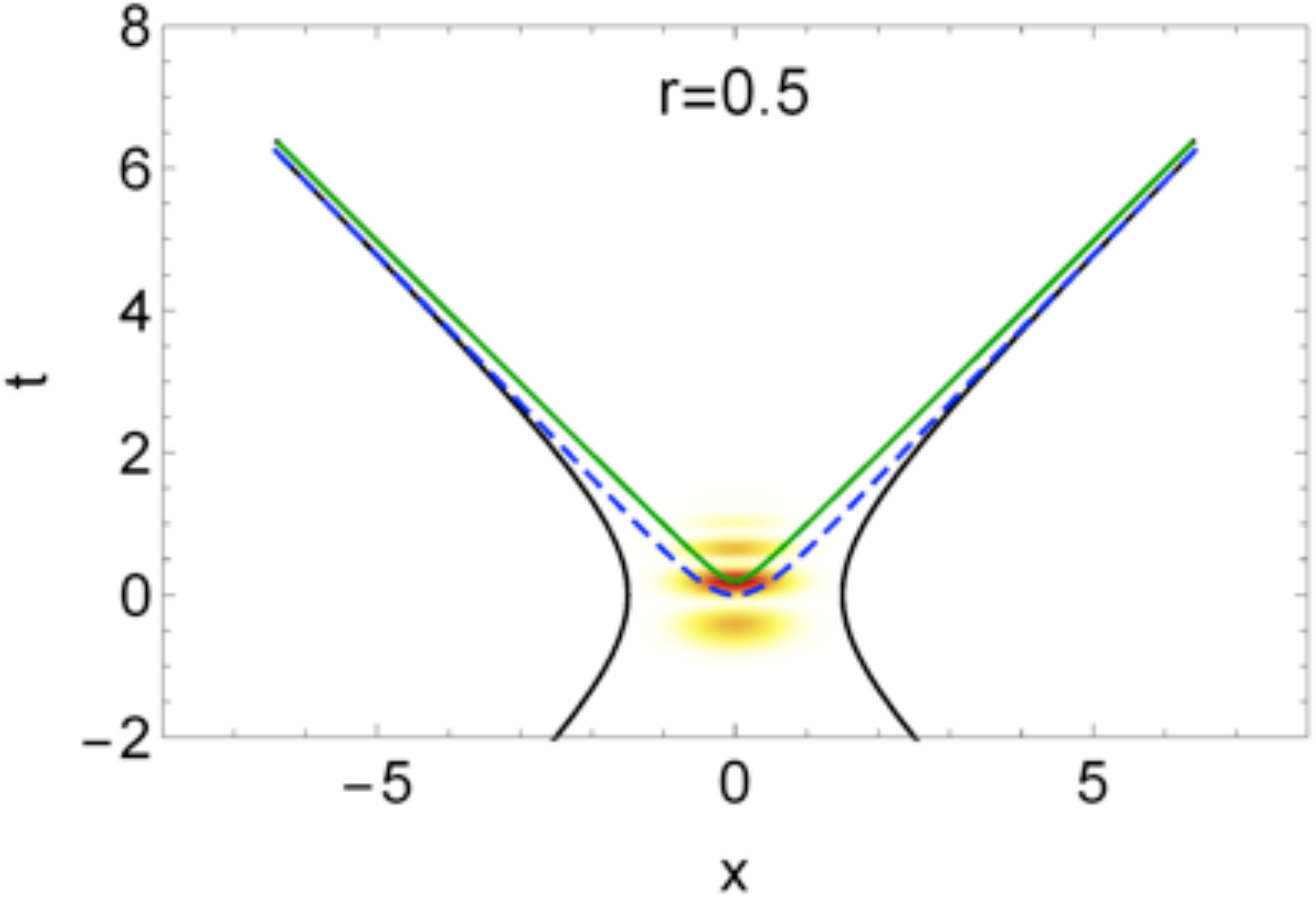
- For the Schwinger effect



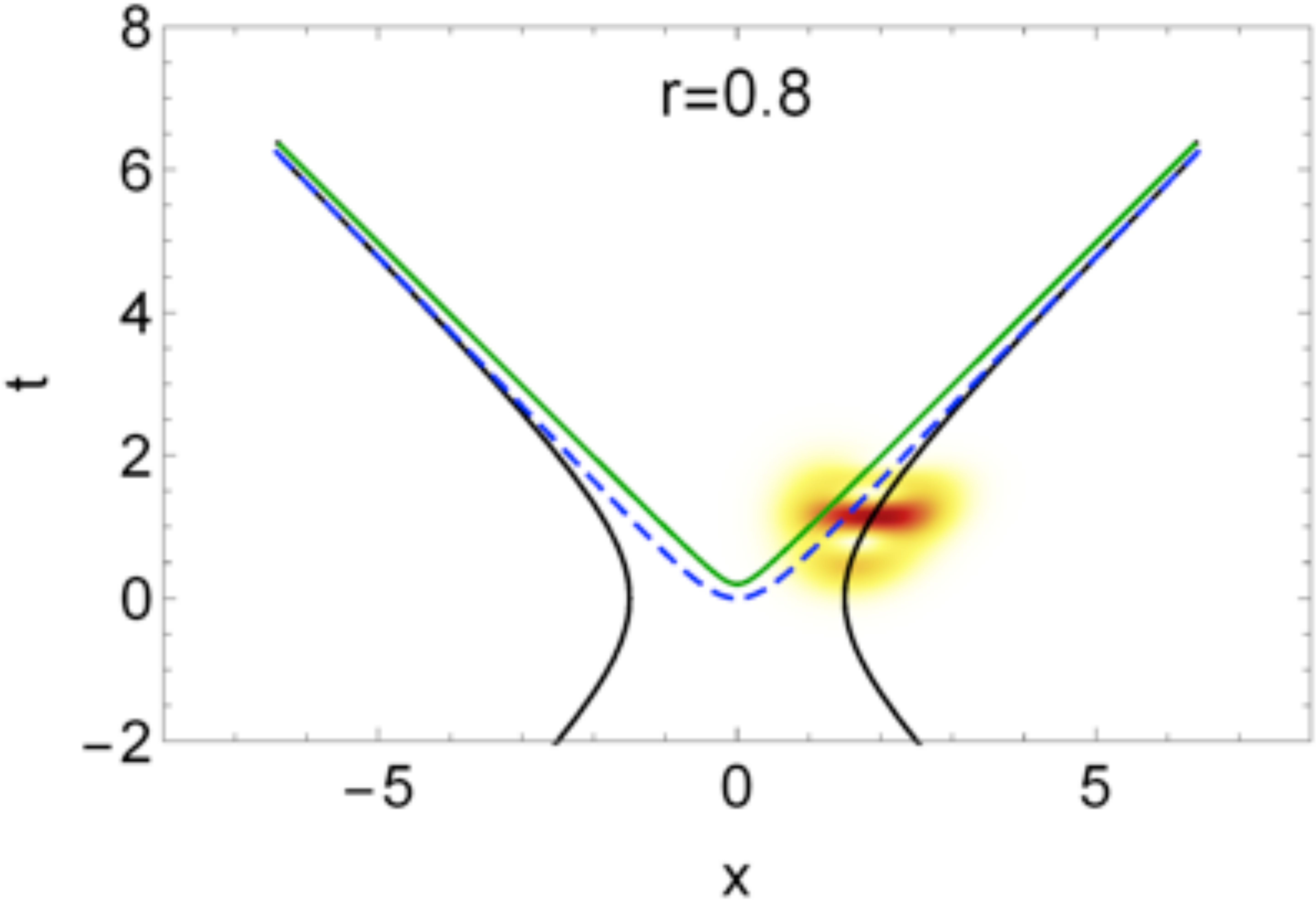
- For the Schwinger effect



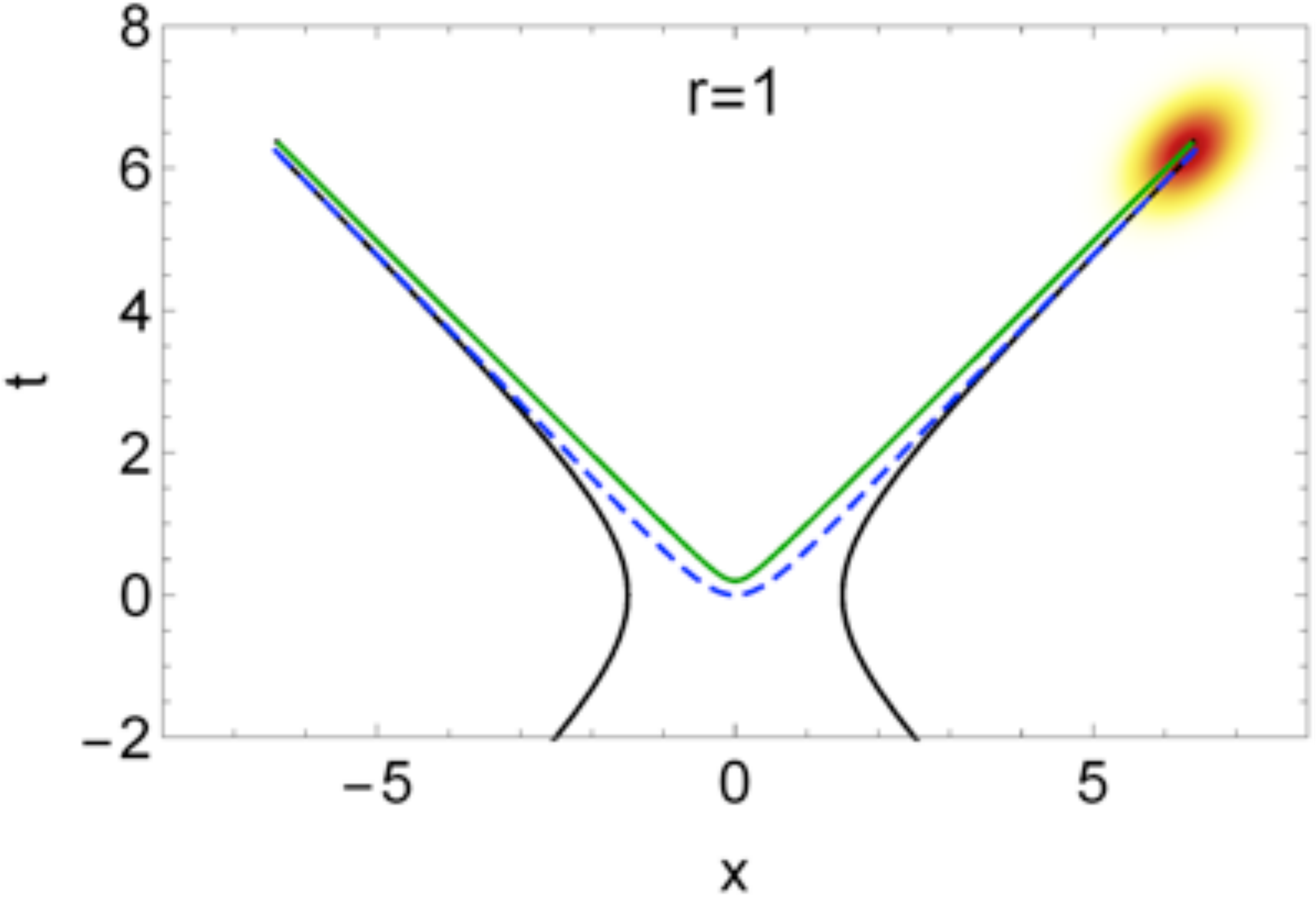
- For the Schwinger effect



- For the Schwinger effect



- For the Schwinger effect



- The weak charge density

$$\rho_w(x^\mu_m) = \frac{e}{m}\,\frac{\int_0^1 \mathrm{d}r \int_{0^+}^{\infty} \mathrm{d}s \int_{\psi_0}^{\psi_1} \mathcal{D}x^\mu e^{iS/\hbar}\,[P_0, \delta(x^\mu(r) - x^\mu_m)]}{\int_{0^+}^{\infty} \mathrm{d}s \int_{\psi_0}^{\psi_1} \mathcal{D}x^\mu e^{iS/\hbar}}$$