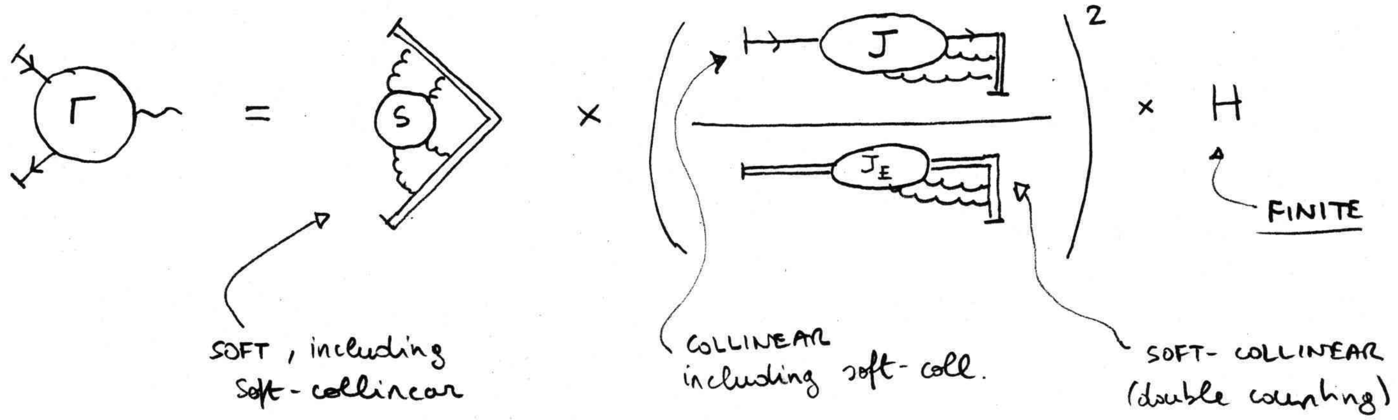
LECTURE 4 : RESUMMATION

① Factorization for the quark form factor.

With the tools described in the previous lectures, one can prove that indeed all soft and collinear divergences in massless gauge theory form factors can be FACTORED into gauge-invariant operator matrix elements. Diagrammatically



The precise definitions are

$$S = S\left(\beta_1 \cdot \beta_2 , \alpha_s(\mu^2), \varepsilon\right) \equiv \langle 0 | \, \Phi_{\beta_2}(\infty, 0) \, \Phi_{\beta_1}(0, -\infty) \, | 0 \rangle$$

$$J = J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \varepsilon\right) \equiv \langle 0 | \, \Phi_n(\infty, 0) \, \psi(0) \, | p \rangle$$

($n^2 \neq 0$, e.g. $n_1^\mu = \beta_1^\mu - \beta_2^\mu = - n_2^\mu$ (not necessary); slight differences for antiparticle or final state, but same scalar function)

$$J_E = J_E\left(\frac{(\beta \cdot n)^2}{n^2}, \alpha_s(\mu^2), \varepsilon\right) \equiv \langle 0 | \, \Phi_n(\infty, 0) \, \Phi_\beta(0, -\infty) \, | 0 \rangle$$

$$H = H\left(\frac{Q^2}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \varepsilon\right)$$

+ FINITE as $\varepsilon \to 0$
+ must cancel (finite) n_i dependence

Then $$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = S \cdot \prod_{i=1}^{2} \frac{J_i}{J_{E,i}} \cdot H$$

COMMENTS

I] The n_i^μ vectors play a three-fold role:

+ ensure gauge invariance
+ replace the opposite-moving hard parton (light-like) with a space-like "absorber" free of collinear div's.
+ act as "FACTORIZATION VECTORS" distinguish COLLINEAR ($p \cdot k < n \cdot k$) from (soft) wide-angle.

II] Functional dependences reflect dimensionality AND homogeneity of eikonal Feynman rules w.r.t. n^μ ($n^2 \neq 0$)

NOTE: homogeneity w.r.t β^μ ($\beta^2 = 0$) is broken by the ANOMALY due to the collinear div. in the UV counterterm. This is precisely determined by the CUSP anomalous dimension.

III] One can construct SINGLE POLE combinations where the "cusp anomaly" CANCELS. In such combinations rescaling invariance under $\beta_i \to \kappa \beta_i$ must be re-established.

+ $\dfrac{J_i}{J_{E,i}}$: only "hard collinear" poles ; no explicit β_i dep.

$$\left(\ln \frac{(p \cdot n)^2}{n^2 \mu^2} - \ln \frac{(\beta \cdot n)^2}{n^2} \sim \ln \frac{Q^2}{\mu^2} \right)$$

+ $$\hat{S} \equiv \frac{S}{J_{E,1} J_{E,2}} = \hat{S}\left(\rho_{12}, \alpha_s(\mu^2), \varepsilon\right), \quad \rho_{12} \equiv \frac{(\beta_1 \cdot \beta_2)^2}{\frac{(\beta_1 \cdot n_1)^2}{n_1^2} \frac{(\beta_2 \cdot n_2)^2}{n_2^2}}$$

$\hookrightarrow$ only "soft wide-angle" poles.

The generalization of this "anomaly cancellation" to multi-leg amplitudes is very powerful...

IV] Whenever a factorization theorem is established (in a renormalizable QFT), a resummation of logarithms follows. Typically they are logarithms of a "factorization scale" (μ, μ_F) arbitrarily introduced. Here we must use independence of Γ on the "factorization vectors" n_i^μ.

(2) From factorization to resummation.

a) Renormalization (it's a factorization too...)

Proving renormalizability is HARD. Then we have a FACTORIZATION THEOREM for Green functions

$$G_0^{(n)}(p_i, \Lambda, g_0) = \prod_{i=1}^{n} Z_i^{1/2}\left(\frac{\Lambda}{\mu}, g(\mu)\right) \cdot G_R^{(n)}(p_i, \mu, g(\mu))$$

⇒ the physics at the cutoff scale enters the problem in a SIMPLE WAY, only through MULTIPLICATIVE CONSTANTS

⇒ We introduced an ARBITRARY SCALE $\mu = \mu_R$ $(p_i \cdot p_j < \mu_R^2 < \Lambda^2)$

The hard work is DONE: trivially, evolution follows from SEPARATION OF VARIABLES

$$\frac{dG_0^{(n)}}{d\mu} = 0 \quad \Rightarrow \quad \frac{d\log(G_R^{(n)})}{d\ln\mu} = -\sum_{i=1}^{n} \gamma_i(g(\mu))$$

where $\gamma_i(g(\mu)) \equiv \frac{1}{2}\frac{d\ln Z_i}{d\ln\mu}$ cannot depend on Λ NOR on p_i!

⇒ Resummation of $\ln(p_i \cdot p_j/\mu^2)$ follows...

b) Factorization in DIS (and elsewhere)

Proving factorization is HARD. Then we have (in Mellin moments), say,

$$\tilde{F}_2\left(N, \frac{Q^2}{m^2}, \alpha_s(Q^2)\right) = \tilde{C}\left(N, \frac{Q^2}{\mu_F^2}, \alpha_s(Q^2)\right) \cdot \tilde{f}\left(N, \frac{\mu_F^2}{m^2}, \alpha_s(Q^2)\right)$$

⇒ all singular behavior as $m^2 \to 0$ is in $\tilde{f}$

⇒ we introduced an arbitrary scale μ_F, which $\tilde{F}_2$ does not know

Then $$\frac{d\tilde{F}_2}{d\mu_F} = 0 \quad \Rightarrow \quad \frac{d\log\tilde{f}}{d\log\mu_F} = -\gamma_N(\alpha_s(Q^2))$$

where $\gamma_N(\alpha_s(Q^2)) \equiv \frac{d\ln\tilde{C}}{d\ln\mu_F}$ cannot depend on Q^2 NOR on m^2!

⇒ Resummation of COLLINEAR $\log(Q^2/\mu_F^2)$ follows...

c) Sudakov resummation.

Double logarithms require a double factorization, and we have just that (Hard/Soft/Collinear).

In fact, we have two sets of equations!

⊕ Renormalization group.

Write
$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = \hat{S}\left(\rho_{12}, \alpha_s(\mu^2), \varepsilon\right) \cdot \prod_{i=1}^{2} J_i\left(\frac{(p_i\cdot n_i)^2}{n_i^2\,\mu^2}, \alpha_s(\mu^2), \varepsilon\right) \cdot H\left(\frac{Q^2}{\mu^2}, \frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \varepsilon\right)$$

Defining
$$\mu\frac{d}{d\mu}\ln J \equiv -\gamma_J(\alpha_s) \quad ; \quad \mu\frac{d}{d\mu}\ln\hat{S} \equiv -\underbrace{\gamma_{\hat{S}}}_{\text{FINITE!}}(\rho_{12}, \alpha_s)$$
$$\mu\frac{d}{d\mu}\ln H \equiv -\gamma_H(\rho_{12}, \alpha_s)$$

RG invariance implies that
$$0 = \mu\frac{d}{d\mu}\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = \gamma_{\hat{S}}(\rho_{12}, \alpha_s) + \gamma_H(\rho_{12}, \alpha_s) + 2\gamma_J(\alpha_s)$$

⊕ Sudakov factorization

Define $x_i \equiv (\beta_i\cdot n_i)^2/n_i^2$ and impose that Γ be independent of n_i^μ (and thus on x_i).

$$x_i\frac{\partial}{\partial x_i}\ln\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = 0 \Rightarrow$$

$$\Rightarrow x_i\frac{\partial}{\partial x_i}\ln J_i = -x_i\frac{\partial}{\partial x_i}\ln H + x_i\frac{\partial}{\partial x_i}\ln J_{E,i} \equiv$$
$$\equiv \frac{1}{2}\left[\mathcal{G}_i\left(x_i, \alpha_s(\mu^2), \varepsilon\right) + \mathcal{K}\left(\alpha_s(\mu^2), \varepsilon\right)\right]$$

the classic "K+G" equation.... (Collins, Soper NPB 193 (1981)).

⇒ $\mathcal{G}$ carries the kinematic dependence BUT IS FINITE as $\varepsilon \to 0$

⇒ $\mathcal{K}$ diverges as $\varepsilon \to 0$ but is INDEPENDENT of KINEMATICS!

(J_{E_i} like S, is a PURE COUNTERTERM, can depend on x_i only through γ_K)

This is sufficient to show that the Q^2 dependence of the FULL FORM FACTOR is ALSO given by a K+G equation. Indeed

$$Q^2 \frac{\partial}{\partial Q^2} \ln \Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = Q^2 \frac{\partial}{\partial Q^2} \ln H + \sum_{i=1}^{2} Q^2 \frac{\partial}{\partial Q^2} \ln J_i =$$

$$= -\frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln H + \sum_{i=1}^{2} x_i \frac{\partial}{\partial x_i} \ln J_i =$$

$$\hookrightarrow \sum_i (G_i + K)$$

$$\hookrightarrow -\frac{1}{2} \mu \frac{d}{d\mu} \ln H + \frac{1}{2} \beta(\varepsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \ln H =$$

$$\hookrightarrow \frac{1}{2} \gamma_H = -\gamma_J - \frac{1}{2} \gamma_{\bar{S}}$$

$$= \underbrace{\frac{1}{2} \beta(\varepsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \ln H - \frac{1}{2} \gamma_{\bar{S}} - \gamma_J + \sum_{i=1}^{2} G_i}_{\frac{1}{2} G} + \underbrace{2K}_{\frac{1}{2} K}$$

Thus

$$Q^2 \frac{\partial}{\partial Q^2} \log\left[\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right)\right] = \frac{1}{2}\left[K(\varepsilon, \alpha_s(\mu^2)) + G\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right)\right]$$

pure counterterm $\qquad$ finite as $\varepsilon \to 0$.

Furthermore

$\mu \frac{d}{d\mu} \Gamma = 0$ implies that G and K must renormalize ADDITIVELY with THE SAME anomalous dimension

$$\mu \frac{d}{d\mu} G\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = -\mu \frac{d}{d\mu} K(\varepsilon, \alpha_s(\mu^2)) = -\beta(\varepsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} K(\varepsilon, \alpha_s) \equiv$$

$$\equiv \gamma_K(\alpha_s(\mu^2))$$ it is the CUSP ANOMALOUS DIMENSION

Now dim. reg. comes into its own. The (K+G) evolution equation has a TRIVIAL INITIAL CONDITION at $Q^2 = 0$

$$\Gamma(0, \alpha_s(\mu^2), \varepsilon) = \Gamma(1, \bar{\alpha}(0, \varepsilon), \varepsilon) = 1.$$

The solution for the form factor is then an exponential with prefactor = 1 :

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left[\frac{1}{2}\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\left(K(\epsilon, \alpha_s(\mu^2)) + G(-1, \bar{\alpha}(\xi^2, \epsilon), \epsilon) + \right.\right.$$
$$\left.\left. + \frac{1}{2}\int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma_K(\bar{\alpha}(\lambda^2, \epsilon))\right)\right]$$

where ⊕ The RG eqn. for G was implemented

⊕ Integration to $-Q^2$ emphasizes that Γ (and G) are real for NEGATIVE Q^2.

$\boxed{\text{NOTE}}$ There is an explicit singularity $K \cdot \int_0 \frac{d\xi^2}{\xi^2}$, which MUST BE CANCELLED by $\int^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma_K$. This can be made explicit by solving the RG eqn. for K :

$$\mu \frac{d}{d\mu} K = \beta \frac{\partial}{\partial \alpha_s} K = -\gamma_K \quad ; \quad K(\epsilon, \alpha_s(0)) = 0 \Rightarrow$$

$$\Rightarrow K(\epsilon, \alpha_s(\mu^2)) = -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma_K(\bar{\alpha}(\lambda^2, \epsilon))$$

Substituting one easily finds (exchanging order of integrations)

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left[\frac{1}{2}\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\left(G(-1, \bar{\alpha}(\xi^2, \epsilon), \epsilon) - \right.\right.$$
$$\left.\left. - \frac{1}{2}\gamma_K(\bar{\alpha}(\xi^2, \epsilon)) \ln\left(\frac{-Q^2}{\xi^2}\right)\right)\right]$$

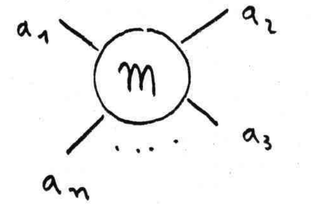
$\boxed{\text{FEATURES}}$

- <u>All</u> IR/C singularities are generated by integrating the coupling.
- <u>Direct</u> contact with finite order calculations in dim. reg.
- <u>Easy</u> analytic continuation $\Gamma(Q^2) \to \Gamma(-Q^2)$, source of π^2's ...
- <u>Closed form</u> results for conformal gauge theories ($N=4$ SYM) where $\beta(\epsilon, \alpha_s) = -2\epsilon\alpha_s$.
- <u>Map</u> to Thomas Becher notation through $\frac{1}{2}\frac{d\mu^2}{\mu^2} = \frac{d\alpha_s}{\beta(\epsilon, \alpha_s)}$.

(3) Multiparton amplitudes

a) Color structures

Multiparton amplitudes can be organized as VECTORS in the space of available color structures



$$= m^{a_1 \cdots a_n} = \sum_L m_L \cdot c_L^{a_1 \cdots a_n}$$

$\boxed{\text{e.g.:}}$ $q\bar{q}$ scattering, tree level 

using $(T^a)_{ij} (T^a)_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right)$

corresponding to color flows , one writes

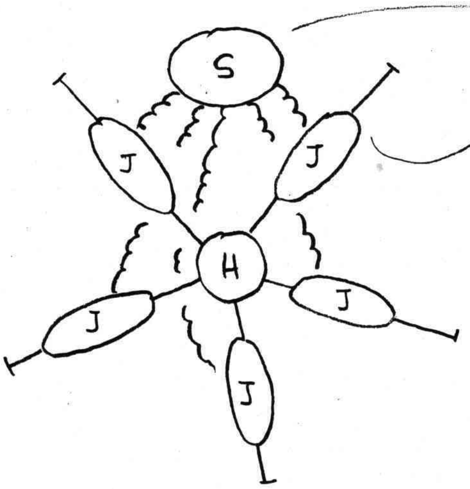
$$m^{ijkl} = m_1 \, c_1^{ijkl} + m_2 \, c_2^{ijkl}$$

with $c_1^{ijkl} = \delta^{il} \delta^{kj}$, $c_2^{ijkl} = \delta^{ij} \delta^{kl}$.

NOTE: exchange of a further gluon (even very soft) MIXES the two color structures!

b) Leading regions

Singularities for FIXED-ANGLE amplitudes ($p_i \cdot p_j \sim Q^2 \gg \Lambda_{QCD}^2 \; \forall i,j$) come from momentum space regions of the form



→ a MATRIX in color space

→ color singlet evolution of parton i can be DEFINED as $\sqrt{\Gamma}$

NOTE: In the PLANAR limit soft gluon exchanges are confined to WEDGES: S can be reabsorbed into the jets and one gets

$$m \Big|_{\text{SING}} \sim \prod_i \Gamma(p_i \cdot p_{i+1})$$

$\Rightarrow$ as a MATRIX, $S \propto \mathbb{1}$ at large N_c.

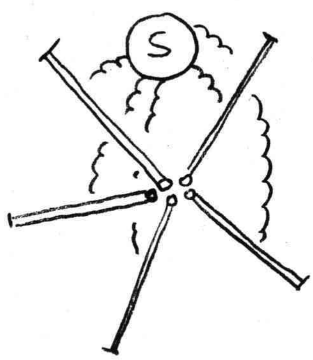
c) Factorization.

$$\mathcal{M}_L\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = S_{LK}\left(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon\right) \cdot H_K\left(\frac{p_i\cdot p_j}{\mu^2}, \frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$
$$\cdot \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ J_{E,i}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right]$$

⇨ S and J_E, as above, are PURE COUNTERTERMS, and acquire inhomogeneous β dependence only through the CUSP. With $n \geq 4$ partons however S can have HOMOGENEOUS β dependence through ratios

$$\rho_{ijkl} \equiv \frac{\beta_i\cdot\beta_j \;\; \beta_k\cdot\beta_l}{\beta_i\cdot\beta_k \;\; \beta_j\cdot\beta_l}$$

d) The soft matrix.



⊕ S is defined as a correlator of n eikonal lines

$$S \equiv \langle 0 | \prod_{i=1}^{n} \overline{\Phi}_{\beta_i}(\infty, 0) | 0 \rangle$$

⊕ S obeys a matrix RGE

$$\mu\frac{d}{d\mu} S_{LK}\left(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon\right) = - S_{LM}(\ldots)\, \Gamma^{(S)}_{MK}\left(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon\right)$$

contains collinear sing.'s

with general solution

$$S(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon) = P\exp\left[-\frac{1}{2}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\,\Gamma^{(S)}\left(\beta_i\cdot\beta_j, \bar{\alpha}(\xi^2,\epsilon), \epsilon\right)\right]$$

⊕ It's easy to define a REDUCED SOFT MATRIX with only single (soft wide-angle) poles and a FINITE anomalous dim. matrix

$$\overline{S}_{LK}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) \equiv \frac{S_{LK}\left(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{n} J_{E,i}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}$$

$$\frac{(\beta_i\cdot\beta_j)^2 n_i^2 n_j^2}{(\beta_i\cdot n_i)^2(\beta_j\cdot n_j)^2}$$

⇨ $$\mu\frac{d}{d\mu}\overline{S}_{LK}(\ldots) = -\overline{S}_{LM}(\ldots)\,\Gamma^{(\bar{S})}_{MK}\left(\rho_{ij}, \alpha_s(\mu^2)\right)$$

↳ FINITE

e) Miracles happen

$$\Gamma^{(\bar{s})}_{KL}(\rho_{ij}, \alpha_s(\mu^2)) = \Gamma^{(s)}_{KL}(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon) - \delta_{KL} \sum_{i=1}^{n} \gamma_{J_E}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)$$

⇨ singular terms in $\Gamma^{(s)}$ must be diagonal and proportional to γ_K.

⇨ finite diagonal terms must conspire to construct ρ_{ij}'s.

⇨ off-diagonal terms in $\Gamma^{(s)}$ must be FINITE and depend only on $\rho_{ijk\ell}$'s.

Indeed, just by FACTORIZATION and the CHAIN RULE, one finds a set of EXACT equations for the ρ dependence of $\Gamma^{(\bar{s})}$:

$$x_i \frac{\partial}{\partial x_i} \Gamma^{(\bar{s})}_{KL} = -\delta_{KL}\, x_i \frac{\partial}{\partial x_i} \gamma_{J_E} = -\frac{1}{4}\gamma_K(\alpha_s)\,\delta_{KL}$$

$$\Rightarrow \sum_{j\neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma^{(\bar{s})}_{KL}(\rho_{ij}, \alpha_s) = \frac{1}{4}\gamma_K(\alpha_s)\,\delta_{KL} \qquad \forall i \qquad (*)$$

A remarkable correlation of color and kinematics!

f) A dipole formula

It's easy to find a solution of the (inhomogeneous) eqn. above.

⇨ Instead of choosing a basis $c_L^{a_1 \dots a_n}$, express $\Gamma^{(s)}$ formally in terms of products of color generators T_i in the repr. of parton i. With appropriate sign conventions they satisfy $\sum_i T_i = 0$.

⇨ Assume that, for partons in rep. R

$$\gamma_K^{(R)}(\alpha_s) = C_R^{(2)}\, \hat{\gamma}_K(\alpha_s)$$

universal, rep.-independent

quadratic Casimir of R

$C_R^{(2)} = T_i \cdot T_i$ (for $i \in R$).

Then

$$\Gamma^{(s)}_{\text{DIP.}}(\rho_{ij}, \alpha_s) = -\frac{1}{8}\hat{\gamma}_K(\alpha_s) \sum_{i\neq j} \ln \rho_{ij}\, T_i\cdot T_j + \frac{1}{2}\hat{\delta}_{\bar{s}}(\alpha_s)\sum_i T_i\cdot T_i$$

solves $(*)$, using $\sum_i T_i = 0$.

Remarkably, only <u>dipole</u> correlations are involved!

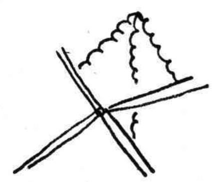
Combining the dipole solution for $\Gamma^{(S)}$ with the solutions for the evolution eqn.'s of partonic jets J_i, and enforcing the cancellation of n_i dependence, one can incorporate ALL IR/C singularities of multiparton amplitudes in a single anomalous dimension matrix

$$\Gamma_{\mathrm{DIP}}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = -\frac{1}{4}\,\hat{\gamma}_K(\alpha_s(\lambda^2)) \sum_{i \neq j} \ln\left(\frac{-2 p_i \cdot p_j}{\lambda^2}\right) T_i \cdot T_j + \sum_{i=1}^{n} \gamma_{J_i}(\alpha_s(\lambda^2))$$

⇨ EXACT up to two loops for any n and to all loops for $n \leq 3$.

⇨ REPRODUCES all known finite order results

⇨ Genuine multiparton correlations absent?



$= 0$ ✓

⇨ Only two kinds of corrections are possible:

a) Presence of higher rank Casimir op's in $\gamma_K^{(R)}(\alpha_s)$.

b) Addition of solutions of the homogeneous equations

$$\sum_{j \neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Delta^{(S)} = 0 \quad ⇨ \quad \Delta^{(S)} = \Delta^{(S)}(\rho_{ijkl})$$

▷ can arise starting at 4 loops (but there are arguments against that, see Thomas' talk)

▷ can arise starting at 3 loops, 4 legs.

Studies are ongoing ...

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