LECTURE 1 : BASICS

1 Qualitative discussion.

FACT In quantum field theory, Two kinds of divergences are associated with the presence of MASSLESS particles.

a) INFRARED (IR, SOFT...) : emission or exchange of particles with vanishing four-momentum ($\lambda_{DB} \to \infty$)

⊕ associated with massless vector bosons : arise in gauge theories ONLY.

⊕ present also when matter particles are MASSIVE.

b) COLLINEAR (C) : splitting of massless particles into parallel moving pairs

⊕ present in any QFT with interaction vertices involving massless particles ONLY.

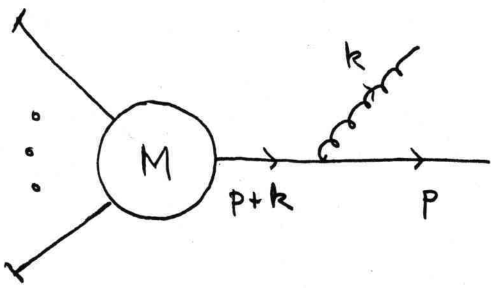
ORIGIN Physical processes happening at LARGE DISTANCES

(Recall : in formal derivation of Feymann rules we integrate $\int d^4x$ over the space-time position of interaction vertices ....)

THERAPY MEASURABLE quantities are NOT AFFECTED by IR/C divergences.

- Long-distance effects are associated with experimentally indistinguishable configurations (energy – angle)
- Must SUM over undistinguishable configuration (by the rules of QM)

AN EXAMPLE Emission of a massless gauge boson.



$$= -ig\, \bar{u}(p)\, \not\varepsilon(k)\, T_a \frac{i(\not p + \not k)}{(p+k)^2 + i\varepsilon} M$$

Possible singularities : $2p\cdot k = 2p_0 k_0 (1-\cos\theta_{pk}) = 0$

- ⊕ $k_0 \to 0$: infrared (soft)
- ⊕ $\cos\theta_{pk} \to 1$: collinear
- ⊕ $p_0 \to 0$: always leads to INTEGRABLE singularities ($u(p) \sim \sqrt{p_0}$).

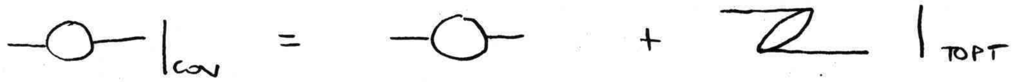
Two mathematical descriptions of what is happening.

a) COVARIANT perturbation theory (as above)

- ⊕ four-momentum is conserved in EVERY VERTEX
- ⊕ price to pay : intermediate particles are generally OFF-SHELL
- ⊕ In IR/C configurations : the emitting fermion is on shell, it can propagate INDEFINITELY before emitting.

b) TIME-ORDERED perturbation theory

- ⊕ From Feynman diagrams :
  - express $\delta(\Sigma E_i)$ at the vertices as Fourier transforms
  - organize "time" integrals into different time orderings

    e.g. : 

  - perform energy integrals to get energy denominators $\left(\sum_i E_i^{(IN)} - \sum_i E_i^{(OUT)}\right)^{-1}$ instead of propagators.
- ⊕ all particles (including intermediate ones) are ON SHELL
- ⊕ price to pay : energy is NOT CONSERVED at vertices
- ⊕ In IR/C configurations : energy IS conserved, the vertex can be ANYWHERE

DIAGNOSIS It's a <u>serious sickness</u>, furthermore at the end of the spectrum that we know well (unlike UV...).

<u>Indeed</u>:

The S-matrix DOES NOT EXIST in Fock space.

$$\langle p_k^{(f)} | S | p_j^{(i)} \rangle \text{ "} = \infty \text{"} \qquad \text{(order by order)}$$

for ANY pair of Fock states, because of LONG RANGE INTERACTIONS (recall: $V(r) \propto 1/r$ ...)

<u>However</u>: Transition probabilities between Fock states are NOT OBSERVABLE

$$| \langle p_k^{(f)} | S | p_j^{(i)} \rangle |^2 \text{ "} = 0 \text{"} \qquad \text{(exactly)}$$

again: because of LONG RANGE INTERACTIONS, asymptotic states are NOT made by isolated charged particles.
... AND: all detectors have finite resolutions in energy and angle!

THERAPY

a) Construct OBSERVABLE transition probabilities.
They are FINITE (KLN theorem)

b) Construct the true asymptotic states (or a decent approximation)
The S matrix between the true asymptotic states is FINITE.

a) KLN theorem (T. D. Lee & M. Nauenberg, PR 133 (1964)
T. Kinoshita, J. Math Phys. 3 (1962))

Consider any quantum theory defined by its hamiltonian H. Let $D_\varepsilon(E_0)$ be the set of exact eigenstates of H with energies $E_0 - \varepsilon \leq E \leq E_0 + \varepsilon$. Let $P(i \to j)$ be the transition probability per unit volume and time between eigenstates i and j.

Then
$$P(E_0, \varepsilon) \equiv \sum_{i,j \in D_\varepsilon(E_0)} P(i \to j)$$
is finite in the limit $m \to 0$ to all orders in perturbation theory.

Notes on KLN theorem.

⊕ In asymptotically free theories the limit $m \to 0$ and the high-energy limit coincide ($\lim_{\mu^2 \to \infty} m(\mu^2) = 0$).

⊕ In ABELIAN theories with $m_f \neq 0$ (QED) a sum over degenerate FINAL states only suffices. (F. Bloch and A. Nordsieck PR 52 (1937) (!))

⊕ In PQCD the KLN theorem is true (of course) but not used in practice when summing over initial states is necessary.

⇒ Hadrons are "given", we do not know the relative weights of parton Fock states in a hadron, but we do know that they are NOT well approximated by the perturbative expansion.

⇒ We will need to resort to other strategies ....

INFRARED SAFETY / FACTORIZATION.

b) The true asymptotic states can be perturbatively approximated by COHERENT STATES, which deserve a section of their own.

2) Coherent states.

$\boxed{\text{BASIC IDEA}}$ Consider the evolution operator for Fock states, in the interaction picture. Formally

$$\Omega_{\pm} \equiv T \exp\left[-i \int_{\mp\infty}^{0} dt \; H(t) \right]$$

interaction Ham. in the "interaction picture"

⇒ "Møller operators", $S = \Omega_-^{\dagger} \Omega_+$

⊕ ISOLATE terms in $H(t)$ that are NOT SUPPRESSED as $t \to \pm\infty$

⊕ CONSTRUCT an ASYMPTOTIC EVOLUTION OPERATOR $\Omega_{\pm}^{(A)}$ with $H \to H^{(A)}$

⊕ DEFINE coherent states as $|h, \pm\rangle \equiv \Omega_{\pm}^{(A)\dagger} |h\rangle$.

The S matrix is then expected to be REGULAR in the coherent state basis (it EXISTS...), since the asymptotic dynamics has been properly taken into account.

$\boxed{\text{FORMALLY}}$

- The evolution operator $U(t,t') \equiv T \exp\left[-i\int_{t'}^{t} dt'' \, H(t'')\right]$

builds the S matrix $S = U(+\infty,0)\, U(0,-\infty) = \Omega_-^\dagger \Omega_+$

- Isolate the asymptotic hamiltonian $H(t) = H_R^E(t) + H_A^E(t)$

(this requires introducing a cutoff, a "factorization energy" E...)

- Extract the asymptotic evolution operator

$$\Omega_{A,\pm}(E) \equiv T \exp\left[-i\int_{\mp\infty}^{0} dt \, H_A^E(t)\right]$$

$$\Omega_\pm = \Omega_{R,\pm}(E)\, \Omega_{A,\pm}(E)$$

(note: H_R and H_A do not commute and don't need to; this DEFINES Ω_R).

- Factorize the S matrix

$$S = \Omega_{A,-}^\dagger \Omega_{R,-}^\dagger \Omega_{R,+} \Omega_{A,+} \equiv \Omega_{A,-}^\dagger S_R \Omega_{A,+}$$

(all depend on E)

- <u>Theorem</u>: $S_R(E) \equiv \Omega_{A,-}(E)\, S\, \Omega_{A,+}^\dagger(E)$

is REGULAR in the Fock basis.

Alternatively: S is regular in the COHERENT STATE basis: $|h,\pm\rangle_E \equiv \Omega_{A,\pm}^\dagger(E)\,|h\rangle$

- Note: this is proven for QED (P.P. Kulish and L.D. Faddeev Theor. Math. Phys. 4 (1970))

(where the coh. state space can be shown to separable and Lorentz- and gauge-invariant)

<u>and</u> for QCD (G. Giavarini and G. Marchesini, NPB 296 (1988))

- How is the finiteness achieved in practice?

A SIMPLE EXAMPLE

Consider the toy scalar theory "φ^3" in $d=6$ ($\Rightarrow$ no IR, only C divergences!)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{\lambda}{6}\varphi^3$$

With the standard mode expansion of the scalar field

$$\varphi(\vec{x},t) = \int \widetilde{dk}\left[a(\vec{k})\,e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t} + a^\dagger(k)\,e^{-i\vec{k}\cdot\vec{x}+i\omega(\vec{k})t}\right]$$

where $\widetilde{dk} = \dfrac{d^5k}{(2\pi)^5\,2\omega(\vec{k})}$ and $\omega(\vec{k}) = |\vec{k}|$,

(recall: this is the interaction picture, operators evolve with H_{FREE} ....)

we can construct the interaction Hamiltonian

$$H(t) \equiv \frac{\lambda}{6}\int d^5x\;\varphi^3(\vec{x},t) = \qquad \text{(check as easy exercise)}$$

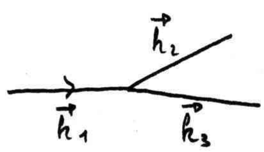
$$= \frac{\lambda}{2}\int \frac{\widetilde{dk}_1\,\widetilde{dk}_2}{2\omega_3(k_1,k_2)}\Big[a^\dagger(\vec{k}_1)\,a(\vec{k}_2)\,a(\vec{k}_1-\vec{k}_2)\,e^{i(\omega_1-\omega_2-\omega_3)t} + \text{h.c.} + (a^\dagger)^3 \text{ terms} + a^3 \text{ terms}\Big]$$

(recognize structure of time orderings $\succ$, $\prec$, $\in$, $\ni$)

Now
- Large-time contributions ($|t|\to\infty$) are SUPPRESSED by oscillating phases UNLESS $\omega_1-\omega_2-\omega_3$ (and similar) VANISHES.
- We can identify the ASYMPTOTIC HAMILTONIAN by inserting a CUTOFF $\Theta(E-|\omega_1-\omega_2-\omega_3|)$

NOTE: terms $\ni$ and $\in$, with $\pm\sum_i\omega_i$, do not contribute!

Parametrize



$$\begin{cases} \vec{k}_1 = \omega_1\,\hat{k}_1 \\ \vec{k}_2 = \omega_1\,(\alpha\,\hat{k}_1 + \beta\,\hat{k}_\perp) \\ \vec{k}_3 = \omega_1\,((1-\alpha)\,\hat{k}_1 - \beta\,\hat{k}_\perp) \end{cases}$$

$$\Rightarrow\quad |\omega_1-\omega_2-\omega_3| = \frac{\beta^2}{2\alpha(1-\alpha)}\,\omega_1 + O(\beta^4)$$

vanishing in the collinear limit $\beta\to 0$

<u>NOTE</u> : anticipating that there will be no IR problem, we have expanded assuming $\beta \ll \alpha$, $\beta \ll 1-\alpha$.

The asymptotic hamiltonian for φ^3 theory is

$$H_A^\Delta(t) = \frac{\lambda}{2} \int \frac{\tilde{dk}_1 \tilde{dk}_2}{2(1-\alpha)\omega_1} \, \theta_\Delta(\vec{k}_1, \vec{k}_2) \left[a^\dagger(\vec{k}_1)\, a(\vec{k}_2)\, a(\vec{k}_1 - \vec{k}_2) \cdot e^{-i \frac{\beta^2}{2\alpha(1-\alpha)} \omega_1 t} + h.c. \right]$$

where θ_Δ is ANY SET of θ functions singling out the collinear limit. E.g. :
$$\theta_\Delta(\vec{k}_1, \vec{k}_2) \equiv \theta(\Delta - \varphi_{k_1 k_2}) \cdot \theta(\Delta - \varphi_{k_2, k_1 - k_2}) .$$

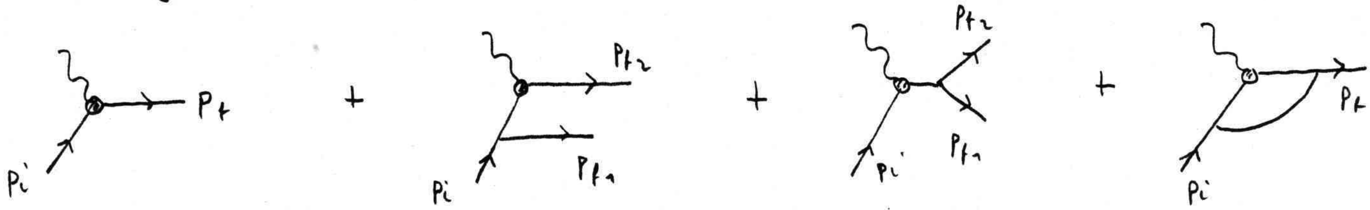
One easily computes the asymptotic evolution operator $\Omega_{A,\pm}(\Delta)$ to first order in λ, integrating over time.

$$\Omega_{A,+}(\Delta) = \frac{\lambda}{2} \int \tilde{dk}_1 \tilde{dk}_2 \, \frac{\alpha}{\beta^2 \omega_1^2} \, \theta_\Delta(\vec{k}_1, \vec{k}_2) \left[a^\dagger(\vec{k}_1)\, a(\vec{k}_2)\, a(\vec{k}_1 - \vec{k}_2) + h.c. \right]$$
$$+ O(\lambda^2) = \Omega_{A,-}(\Delta) + O(\lambda^2) .$$

Each term has a transparent physical interpretation, and the collinear divergence is in evidence.

$\boxed{\text{EXERCISE}}$ Cancellation of collinear divergences in "φ^3" DIS.

Inclusive scattering by a classical source as usual receives corrections of the form



Now we can define REGULAR S-matrix elements SEPARATELY for the real emission and the virtual correction processes. Consider, for real emission

$$\langle p_{f1}, p_{f2} |\, S_R(\Delta)\, | p_i \rangle = \langle 0 |\, a(\vec{p}_{f1})\, a(\vec{p}_{f2})\, \Omega_{A,-}(\Delta)\, S\, \Omega^\dagger_{A,+}(\Delta)\, a^\dagger(p_i) | 0 \rangle$$

Focus, say, on the situation where $\vec{p}_{f2}$ is collinear with $\vec{p}_i$. Then $\Omega_{A,-}(\Delta)$ cannot contribute, since $\vec{p}_{f1}$ and $\vec{p}_{f2}$ are NOT collinear (S injects a nontrivial momentum $\vec{q}$).

There are then 2 contributions

$$\langle p_{f1}, p_{f2} | S_R(\Delta) | p_i \rangle = \langle 0 | a(\vec{p}_{f1})\, a(\vec{p}_{f2})\, S^{(0)}\, \Omega^{\dagger\,(1)}_{A,+}(\Delta)\, a^\dagger(p_i) | 0 \rangle +$$

$$+ \langle 0 | a(\vec{p}_{f1})\, a(\vec{p}_{f2})\, S^{(1)}\, a^\dagger(\vec{p}_i) | 0 \rangle$$

⊕ In the first contribution the coherent state op. $\Omega^\dagger$ creates a state with two coll. particles, one of which is scattered by $S^{(0)}$:

$$\Omega^{\dagger\,(1)}_{A,+}(\Delta)\, a^\dagger(\vec{p}_i) |0\rangle = \frac{\lambda}{2} \int \widetilde{dk}\, \frac{\alpha}{\beta^2 \omega_i^2}\, \theta_\Delta(\vec{p}_i, \vec{k})\, a^\dagger(\vec{k})\, a^\dagger(\vec{p}_i - \vec{k}) |0\rangle$$

Then defining $\langle \bar{p}_f | S | \bar{p}_i \rangle \equiv M_0(\bar{p}_f, \bar{p}_i)$ we get 2 identical contributions identifying $\vec{k}$ (or $\vec{p}_i - \vec{k}$) with $\vec{p}_{f1}$

$$\langle 0 | a\, a\, S^{(0)} \Omega^{(1)} a^\dagger | 0 \rangle = \lambda\, M_0(\bar{p}_{f2}, \bar{p}_i - \bar{p}_{f1})\, \frac{\alpha_{f1}}{(\omega_i \beta_{f1})^2}\, \theta_\Delta(\bar{p}_i, \bar{p}_{f1})$$

⊕ The second contribution corresponds to the standard diagrams  $\longrightarrow M_0(\bar{p}_{f1} + \bar{p}_{f2}, \bar{p}_i) \cdot \frac{1}{(\bar{p}_{f1} + \bar{p}_{f2})^2}$

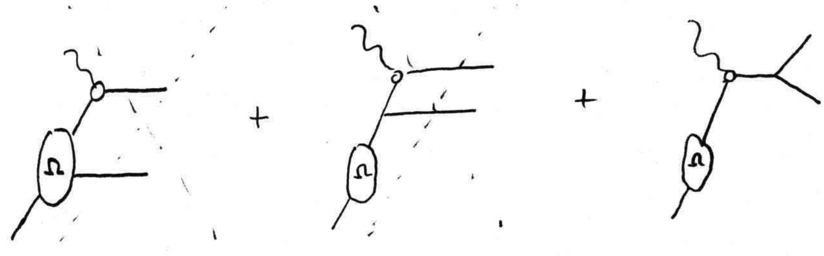
$\longrightarrow \lambda\, M_0(\bar{p}_{f2}, \bar{p}_i - \bar{p}_{f1}) \left(- \frac{\alpha_{f1}}{\beta_{f1}^2 \omega_i^2} \right)$

in the collinear limit

The singular contribution of the first diagram is precisely cancelled by the coherent state, acting as a LOCAL COUNTERTERM.

Graphically

$\langle f_1, f_2 | S_R(\Delta) | i \rangle =$



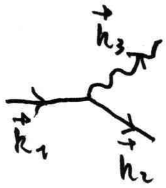
IN GAUGE THEORIES

- The asymptotic hamiltonian has an IR component as well as a collinear one

$$H_A^E(t) = H_{IR}^E(t) + H_c^E(t)$$

NOTE: they overlap! $H_{IR,C}^E(t)$ can be associated with either.

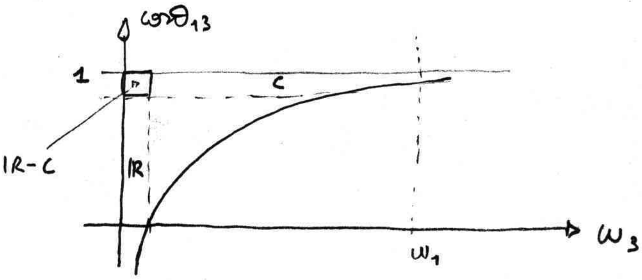
- The asymptotic energy region is defined by



$$|\omega_1 - \omega_2 - \omega_3| < E \qquad (\omega_3 = |\vec{k}_3|)$$

$$\Rightarrow \left| \omega_1 - \omega_3 - \sqrt{\omega_1^2 + \omega_3^2 - 2\omega_1\omega_3 \cos\theta_{13}} \right| < E$$

In the $\cos\theta_{13} - \omega_3$ plane



- In the IR region the asymptotic hamiltonian simplifies considerably: one can neglect the soft momentum inside non-singular functions ....

$$H_{IR}(t) \propto g \int \tilde{dk}\, \tilde{dp}\, \frac{1}{\omega_p}\, \theta(E - \hat{p}\cdot\vec{k})\, \mathcal{H}_{EIK}(\vec{p},\vec{k})$$

where $$\mathcal{H}_{EIK}(\vec{p},\vec{k}) = \rho(\vec{p}) \sum_\lambda \vec{p}\cdot\varepsilon_{(\lambda)}(\vec{k}) \left[a_{(\lambda)}(\vec{k})\, e^{-i\hat{p}\cdot k t} + \text{h.c.} \right]$$

and $$\rho(\vec{p}) = \sum_s \left[b^\dagger_{(s)}(\vec{p})\, b_{(s)}(\vec{p}) - d^\dagger_{(s)}(\vec{p})\, d_{(s)}(\vec{p}) \right]$$

This is the situation in QED, with massive fermions, where we can stay away from collinear limits (so $\hat{p}\cdot\vec{k} \sim |\vec{k}|$).

$\Rightarrow$ Fermion operators only appear in the number density $\rho(\vec{p})$: Fock states with definite # of fermions are eigenstates

$\Rightarrow$ The coherent state is an "ordinary" coherent state: $\sim e^{\alpha a + \alpha^* a^\dagger}$

③ The strategy of PQCD.

a) Coherent states CAN be used to build a subtraction algorithm. They CANNOT be used in PT as approximations to the true asymptotic states (HADRONS)

b) Alternative 1: Infrared Safety

- Compute partonic $\hat{\sigma}$'s with IR regulators
  (e.g.: $\varepsilon = 2 - d/2 < 0$)
  requiring a hard scale Q^2 to use $\alpha_s(Q^2) \ll 1$.

$$\hat{\sigma} = \hat{\sigma}\left(\frac{Q^2}{\mu^2}, \, p_i \cdot p_j, \, \alpha_s(\mu^2); \, \frac{m_i^2(\mu^2)}{\mu^2}, \, \varepsilon\right)$$

- Select quantities with a FINITE $\varepsilon, m \to 0$ limit

$$\hat{\sigma} = \hat{\sigma}\left(\frac{Q^2}{\mu^2}, \, p_i \cdot p_j, \, \alpha_s(\mu^2); \, 0, 0\right) + O\left(\left(\frac{m^2}{\mu^2}\right)^p, \, \varepsilon\right)$$

- Interpret these IR/C safe $\hat{\sigma}$'s (insensitive to long-distance, hadronization effects) as perturbative estimates of the corresponding HADRONIC cross sections
  ("LOCAL PARTON – HADRON DUALITY"),
  valid up to $O((\Lambda_{QCD}^2/Q^2)^p)$ corrections.

c) Alternative 2: Factorization

- With IDENTIFIED HADRONS in the initial OR final state IR/C sensitivity remains.

- Exploit quantum mechanical incoherence in the presence of disparate scales Λ_{QCD}, Q^2 to prove FACTORIZATION THEOREMS at PARTON LEVEL

$$\hat{\sigma} = f\left(\frac{m^2}{\mu_F^2}\right) * \hat{\hat{\sigma}}\left(\frac{Q^2}{\mu^2}, \frac{\mu_F^2}{\mu^2}\right) + O\left(\left(\frac{m^2}{\mu_F^2}\right)^p\right)$$

- Interpret f's as universal, NP, wave function contributions to be MEASURED and combined with process-dep., perturbative $\hat{\hat{\sigma}}$. Note, μ_F dependence must be known...