Wishart Random Matrices, Vicious Walkers and 2-d Yang-Mills Gauge Theory

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Plan

WISHART RANDOM MATRICES

↓ Wishart 1928, Tracy–Widom 1993, Johansson 2000 ....

VICIOUS BROWNIAN WALKERS

↓ de Gennes 1968, Fisher 1984, ...

2–d YANG–MILLS THEORY ON THE SPHERE

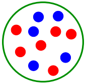
LARGE N PHASE TRANSITION (3rd order)

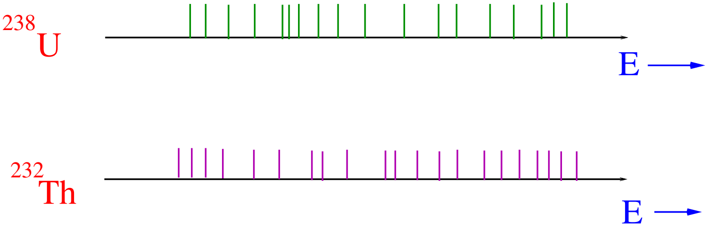
LATTICE (Wilson Action): Gross and Witten 1980, Wadia 1980....

CONTINUUM : Migdal 1975 , Rusakov 1990, Douglas and Kazakov 1993, Gross and Matytsin 1994....

I : Wishart Random Matrices

spectra of heavy nuclei





WIGNER ('50) : replace complex H by random matrix

DYSON, GAUDIN, MEHTA, ...

Applications of Random Matrices

Physics: nuclear physics, quantum chaos, disorder and localization, mesoscopic transport, optics/lasers, quantum entanglement, neural networks, gauge theory, QCD, matrix models, cosmology, string theory, statistical physics (growth models, interface, directed polymers...), ...

Mathematics: Riemann zeta function (number theory), Voiculescu's free probability theory, combinatorics and knot theory, determinantal points processes, integrable systems, ...

Statistics: multivariate statistics, principal component analysis (PCA), image processing, data compression, Bayesian model selection, ...

Information Theory: signal processing, wireless communications, ..

Biology: sequence matching, RNA folding, gene expression network

Economics and Finance: time series analysis,....

Recent Ref: The Oxford Handbook of Random Matrix Theory
ed. by G. Akemann, J. Baik and P. Di Francesco (2011)

First Appearence of Random Matrices

Biometrika, 20, 32-52 (1928)

THE GENERALISED PRODUCT MOMENT DISTRIBUTION IN SAMPLES FROM A NORMAL MULTIVARIATE POPULATION.

By JOHN WISHART, M.A., B.Sc. Statistical Department, Rothamsted Experimental Station.

1. *Introduction.*

For some years prior to 1915, various writers struggled with the problems that arise when samples are taken from uni-variate and bi-variate populations, assumed in most cases for simplicity to be normal. Thus "Student," in 1908*, by considering the first four moments, was led by K. Pearson's methods to infer the distribution of standard deviations, in samples from a normal population. His results, for comparison with others to be deduced later, will be stated in the form

$$dp = \frac{1}{\Gamma\left(\frac{N-1}{\ldots}\right)} A^{\frac{N-1}{2}} \cdot e^{-Aa} \cdot a^{\frac{N-3}{2}} \, da \quad \ldots\ldots (1),$$

John Wishart (1898-1956)

SESSION IIB

INTERPRETATION OF LOW ENERGY NEUTRON SPECTROSCOPY

CHAIRMAN—W. W. Havens, Jr.

IIB1. DISTRIBUTION OF NEUTRON RESONANCE LEVEL SPACING.

E. P. WIGNER, *Princeton University*
Presented by E. P. Wigner

The problem of the spacing of levels is neither a terribly important one nor have I solved it. That is really the point which I want to make very definitely. As we go up in the energy scale it is evident that the detailed analyses which we have seen for low energy levels is not possible, and we can only make

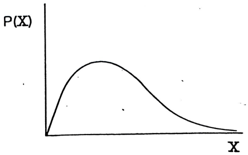


Fig. IIB1-1. Probability of a level spacing X.

›ur, that is a much more serious deviation and much ›ss probable statistically.

Let me say only one more word. It is very likely ıat the curve in Figure I is a universal function. ı other words, it doesn't depend on the details of ıe model with which you are working. There is one articular model in which the probability of the nergy levels can be written down exactly. I mentoned this distribution already in Gatlinburg. It is alled the Wishart distribution. Consider a set of ymmetric matrixes in such a way that the diagonal ement m_{11} has a distribution $\exp(-m_{11}^2/4)$. In other ords, the probability that this diagonal element ıall assume the value m_{11} is proportional to ‹p$(-m_{11}^2/4)$. Then as I mentioned, and this was ıown a long time ago by Wishart, the probability for ıe characteristic roots to be $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$, if ıis is an n dimensional matrix, is given by the ‹pression:

$$-1/2\,(\lambda_1^2 + \lambda_2^2 + \ldots$$

bility that two successive roots have a distance X, then you have to integrate over all of them except two. This is very easy to do for the first integration, possible to do for the second integration, but when you get to the third, fourth and fifth, etc., integrations you have the same problem as in statistical mechanics, and presumably the solution of the problem will be accomplished by one of the methods of statistical mechanics. Let me only mention that I did integrate over all of them except one, and the result is $\frac{1}{2\pi}\sqrt{4n - \lambda^2}$. This is the probability that the root shall be λ. All I have to do is to integrate over one less variable than I have integrated over, but this I have not been able to do so far.

DISCUSSION

W. HAVENS: Where does one find out about a the root shall be λ. All I have to do is to integrate over one less variable than I have integrated over, but this I have not been able to do so far.

DISCUSSION

W. HAVENS: Where does one find out about a Wishart distribution?

E. WIGNER: A Wishart distribution is given in S. S. Wilks book about statistics and I found it just by accident.

Covariance Matrix

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{vmatrix} \text{phys.} & \text{math} \\ X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{33} \end{vmatrix}$$

in general (MxN)

$$X^t = \begin{vmatrix} X_{11} & X_{21} & X_{31} \\ X_{12} & X_{22} & X_{33} \end{vmatrix}$$

in general (NxM)

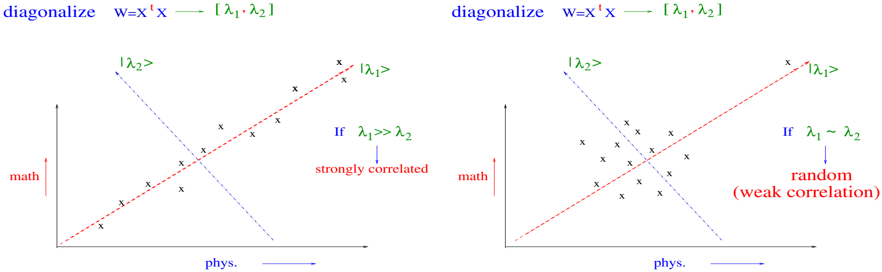
$$W = X^t X = \begin{vmatrix} X_{11}^2 + X_{21}^2 + X_{31}^2 & X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{33} \\ X_{12}X_{11} + X_{22}X_{21} + X_{33}X_{31} & X_{12}^2 + X_{22}^2 + X_{33}^2 \end{vmatrix}$$

→ (NxN) COVARIANCE MATRIX (unnormalized)

Principal Component Analysis

Consider N students and $M = 2$ subjects (phys. and math.)

$X \to (N \times 2)$ matrix and $W = X^t X \to 2 \times 2$ matrix



data compression via 'Principal Component Analysis' (PCA)

$\longrightarrow$ practical method for image compression in computer vision

Null model $\to$ random data: $X \to$ random $(M \times N)$ matrix

$\to W = X^t X \to$ random $N \times N$ matrix (Wishart, 1928)

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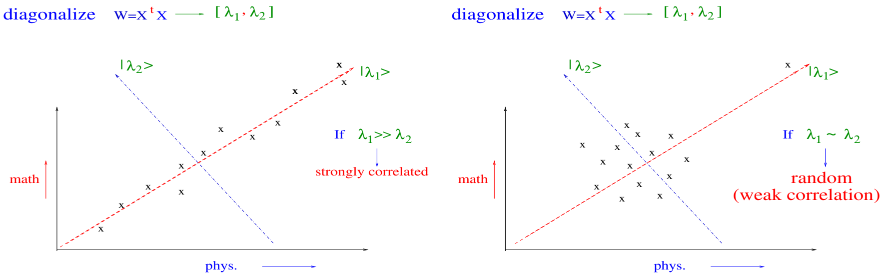

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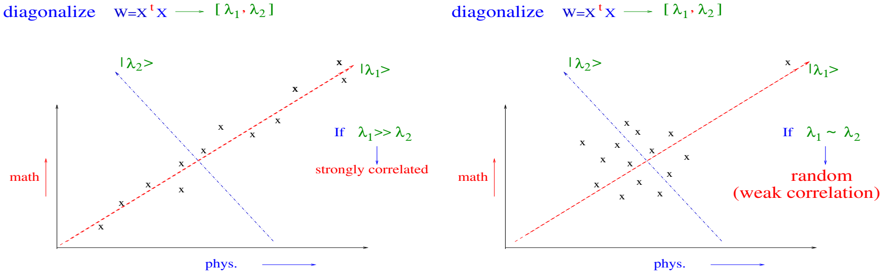

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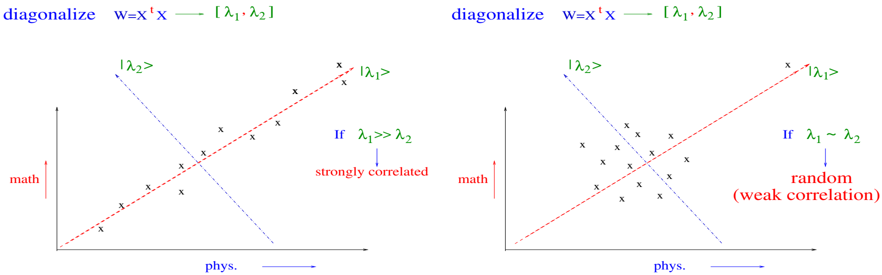

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Random Data: Wishart Random Matrix

- Let $X = [x_{i,j}] \to (M \times N)$ rectangular data matrix

- $W = X^{\dagger}X \to (N \times N)$ square covariance matrix (Wishart)

- Entries of X Gaussian: $\Pr[X] \propto \exp\left[-\frac{\beta}{2}\mathrm{Tr}(X^{\dagger}X)\right]$

  $\beta = 1 \to$ Real entries, $\beta = 2 \to$ Complex

- N real eigenvalues of W: $\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_N \geq 0$

- Joint distribution of eigenvalues (James, 1960):

$$P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

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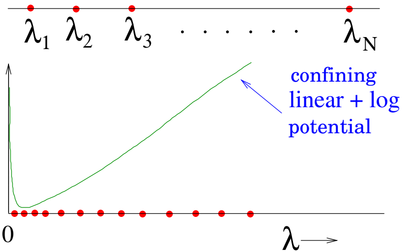
Coulomb Gas interpretation

- $P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N}(\lambda_i - a\log\lambda_i) - \sum_{j\neq k}\log|\lambda_j - \lambda_k|\right\}\right]$

  where $a = M - N + 1 - \frac{2}{\beta}$
- 2-d Coulomb gas confined to a line (Dyson) with $\beta \to$ inverse temp.



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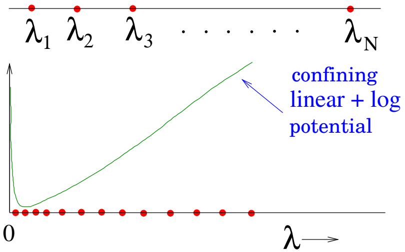

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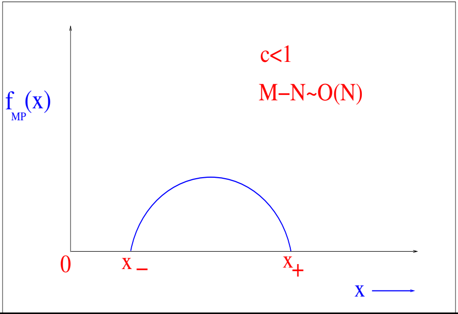
Spectral Density: Marcenko-Pastur Law

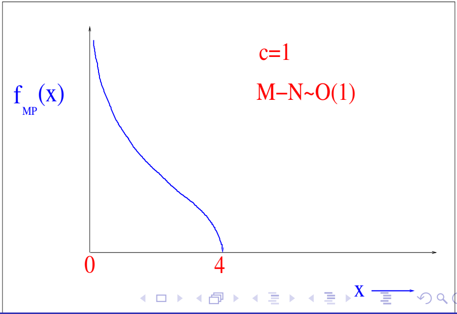
- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle \xrightarrow[N\to\infty]{} \frac{1}{N} f_{\rm MP}(\frac{\lambda}{N})$

- Marcenko-Pastur law (1967): $\boxed{f_{\rm MP}(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}}$

  $x_\pm = (1 \pm \frac{1}{\sqrt{c}})^2$ where $c = N/M \leq 1$

- for $c = 1$ ($M - N \sim O(1)$): $f_{\rm MP}(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$





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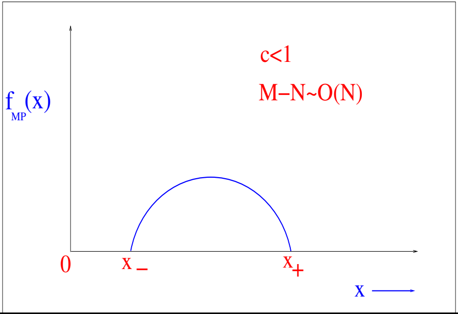
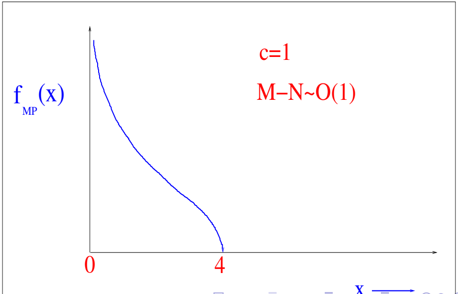
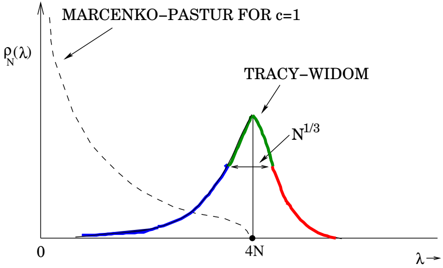




Largest eigenvalue: Tracy-Widom distribution



Largest eigenvalue $\lambda_{\max}$ fluctuates from sample to sample

- $\langle \lambda_{\max} \rangle = 4N$; typical fluctuation: $|\lambda_{\max} - 4N| \sim N^{1/3}$ (small)
- typical fluctuations are distributed via Tracy-Widom, 1994 law

(Johansson 2000, Johnstone, 2001)

- cumulative distribution: $\mathrm{Prob}[\lambda_{\max} \leq t, N] \rightarrow \mathcal{F}_\beta\left(\frac{t-4N}{2^{4/3}N^{1/3}}\right)$

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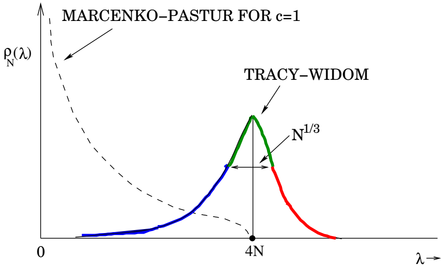

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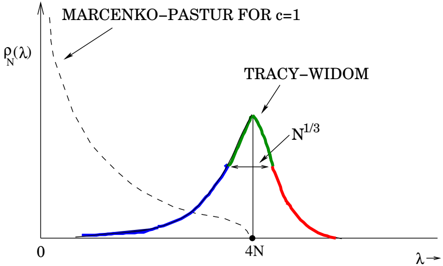

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Tracy-Widom distribution (1994)

The scaling function $\mathcal{F}_\beta(x)$ has the expression:

- $\beta = 1$: $\mathcal{F}_1(x) = \exp\left[-\frac{1}{2}\int_x^\infty \left[(y-x)q^2(y) - q(y)\right]\, dy\right]$
- $\beta = 2$: $\mathcal{F}_2(x) = \exp\left[-\int_x^\infty (y-x)q^2(y)\, dy\right]$

$$\frac{d^2q}{dy^2} = 2\,q^3(y) + yq(y) \rightarrow \text{Painlevé equation}$$

- Note that $\mathcal{F}_\beta(x) \rightarrow$ Cumulative distribution

$$\mathcal{F}_\beta(x) \rightarrow 0 \ (\text{as } x \rightarrow -\infty)$$

$$\mathcal{F}_\beta(x) \rightarrow 1 \ (\text{as } x \rightarrow \infty)$$

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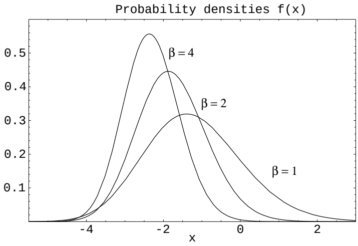
Tracy-Widom distribution for $\lambda_{\max}$



- Tracy-Widom density $f_\beta(x)$ depends explicitly on β.
- Asymptotics: $f_\beta(x) \sim \exp\left[-\frac{\beta}{24}|x|^3\right] \quad \text{as} \quad x \to -\infty$

$$\sim \exp\left[-\frac{2\beta}{3}\, x^{3/2}\right] \quad \text{as} \quad x \to \infty$$

Applications: Growth models, Directed polymer, Sequence Matching, ...

(Baik, Deift, Johansson, Prahofer, Spohn, Johnstone,....)

[S.M., Les Houches lecture notes (2006)]

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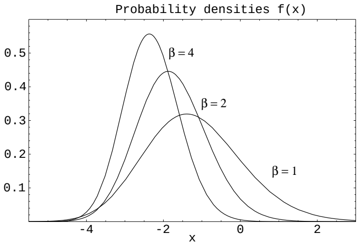

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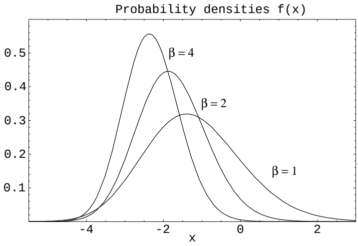


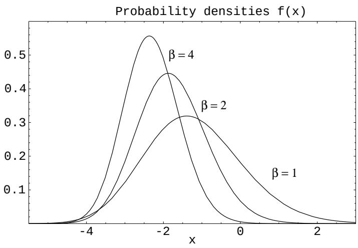
- Tracy-Widom density $f_\beta(x)$ depends explicitly on β.
- Asymptotics: $f_\beta(x) \sim \exp\left[-\frac{\beta}{24}|x|^3\right] \quad \text{as} \quad x \to -\infty$

$$\sim \exp\left[-\frac{2\beta}{3}\, x^{3/2}\right] \quad \text{as} \quad x \to \infty$$

Applications: Growth models, Directed polymer, Sequence Matching, ...

(Baik, Deift, Johansson, Prahofer, Spohn, Johnstone,....)

[S.M., Les Houches lecture notes (2006)]



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Applications: Growth models, Directed polymer, Sequence Matching, ...

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[S.M., Les Houches lecture notes (2006)]

Two facts to remember:

For Wishart matrices with complex entries ($\beta = 2$)

- Joint distribution of eigenvalues:

$$P(\{\lambda_i\}) \propto \exp\left[-\sum_{i=1}^{N} \lambda_i\right] \prod_i \lambda_i^{M-N} \prod_{j<k} |\lambda_j - \lambda_k|^2$$

- Largest eigenvalue $\lambda_{\max}$ (centered and scaled) is distributed via

Tracy-Widom: $\mathcal{F}_2(x)$

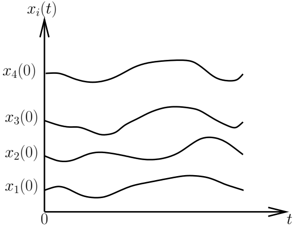
II : Nonintersecting Brownian Motions

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t)\ ,\ \langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{i,j}\delta(t-t')$$
$$x_1(0) < x_2(0) < ... < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < ... < x_N(t)\ ,\quad \forall t \geq 0$$



P.-G. De Gennes, 1968, D. A. Huse and M. E. Fisher, 1984, ...

Fibrous Polymers

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. DE GENNES

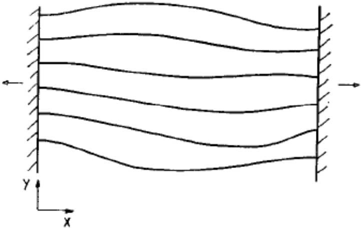


FIG. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

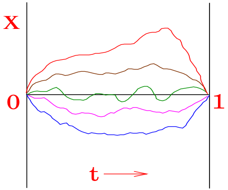
Non-intersecting Brownian bridges in 1d

- N Brownian bridges in one-dimension

$$\dot{x}_i(t) = \zeta_i(t) \ , \quad \langle \zeta_i(t)\zeta_j(t') \rangle = \delta_{i,j}\delta(t-t') \ , \quad 0 \le t \le 1$$
$$x_i(0) = x_i(t=1) = 0$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < \ldots < x_N(t)$$
$$0 < \forall t < 1$$



watermelon

Reunion probability $\rightarrow$ Comm.-Incomm. phase transition

Huse & Fisher, Fisher (1984), ..., Johansson (2002), Ferrari & Praehofer (2006),...

Non-intersecting Brownian excursions in 1d

- N Brownian excursions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t)\ , \quad \langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{i,j}\delta(t-t')\ , \quad 0 \le t \le 1$$

$$x_i(0) = x_i(t=1) = 0 \qquad x_i(t) > 0 \quad \mathit{for} \quad 0 < t < 1$$

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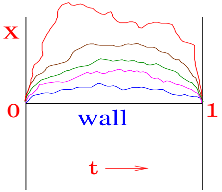
$$x_1(t) < x_2(t) < \ldots < x_N(t)$$

$$0 < \forall t < 1$$



half–watermelon

watermelon "with a wall"

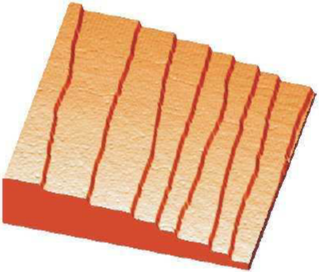
Katori & Tanemura (2004), Tracy & Widom (2007), ...

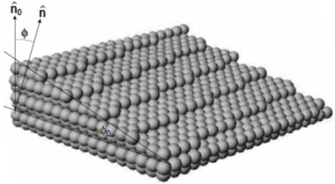
Vicious Walkers in Physics

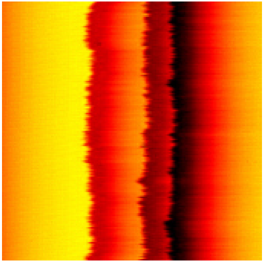
- P. G. de Gennes, *Soluble Models for fibrous structures with steric constraints* (1968)
- D. A. Huse and M. E. Fisher, *Commensurate melting, domain walls, and dislocations (1984)*; M. E. Fisher, *Walks, Walls, Wetting and Melting* (1984)
- B. Duplantier *Statistical Mechanics of Polymer Networks of Any Topology* (1989)
- J. W. Essam, A. J. Guttmann, *Vicious walkers and directed polymer networks in general dimensions* (1995)
- H. Spohn, M. Praehofer, P. L. Ferrari et al. *Stochastic growth models* (2006)
- T. L. Einstein et. al., *Fluctuating step edges on vicinal surfaces* (2004–)
- ...

Connection between Vicious Walkers and Random Matrix Theory

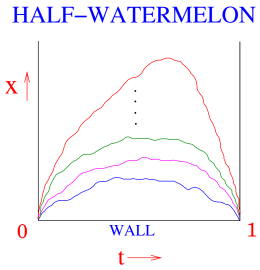
Fluctuating step edges: Maryland group







Brownian excursions and Dyck paths



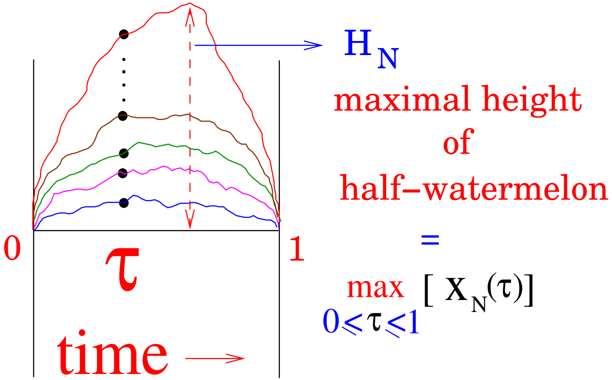
In presence of a hard wall at the origin → half-watermelons

Continuous space-time: Non-intersecting Brownian Excursions

Discrete space-time: Dyck paths (combinatorial objects)

(Cardy, Katori, Tanemura, Krattenthaler, Fulmek, Feierl, Guttmann, Viennot, Tracy-Widom ...)

Two questions



Q1: At fixed $0 < \tau < 1$, what is the joint distribution of positions $P_{\rm joint}(\{x_i\}|\tau)$? ⇒ local question

Q2: What is the probability distribution of the global maximal height $\text{Prob.}[H_N \leq M, N] = F_N(M)$? ⇒ global question

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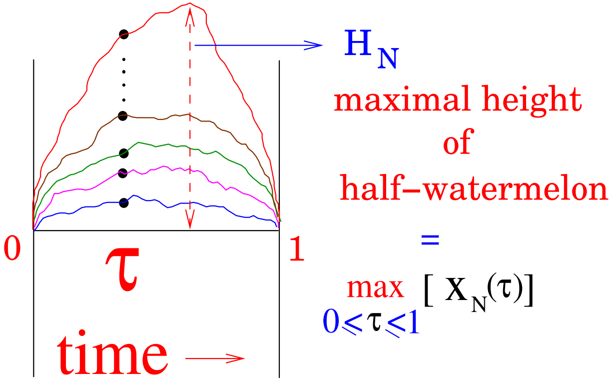

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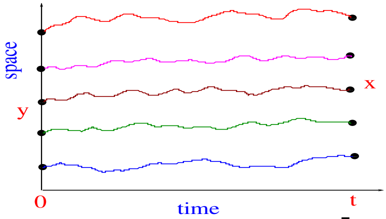
Method: Path Integral for free fermions



Propagator from $\vec{y}$ at $\tau = 0$ to $\vec{x}$ at $\tau = t$

$$G(\vec{x}, \vec{y}, t) = \int_{\vec{y}}^{\vec{x}} \mathcal{D}\vec{x}(\tau)\, \exp\left[-\frac{1}{2}\int_0^t \sum_i \dot{x}_i^2(\tau) d\tau\right] \mathbf{1}_{x_1(\tau)<x_2(\tau)<\cdots<x_N(\tau)}$$

$$= \langle \vec{x}| e^{-\hat{H}t} |\vec{y}\rangle; \text{ where } \hat{H} \equiv -\frac{1}{2}\sum_i \partial_{x_i}^2$$

$$= \sum_E \psi_E(\vec{x})\psi_E(\vec{y})\, e^{-E\,t}$$

$\psi_E(\vec{x}) \equiv \det[\phi_{n_i}(x_j)] \to$ Slater determinant ($N \times N$)

Alternative methods:

- Lindstrom-Gessel-Viennot method (discrete lattice paths)
- Karlin-Mcgregor formula (continuous paths)

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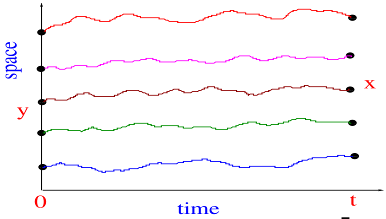

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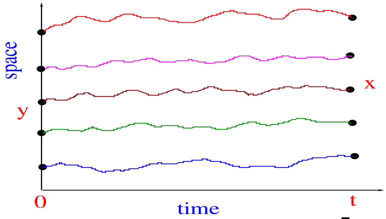

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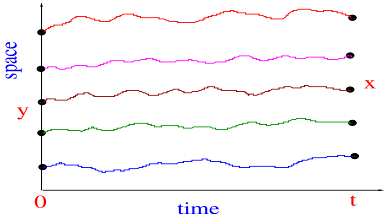

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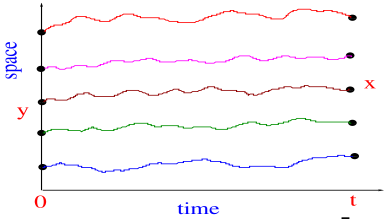

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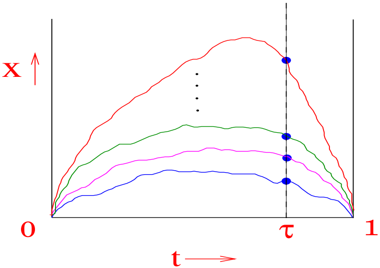
Q1: Joint distribution → Wishart eigenvalues



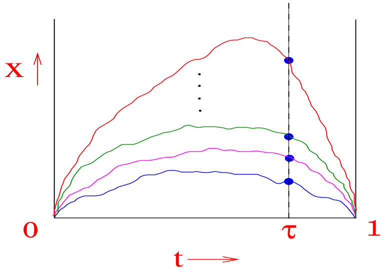
At fixed time $0 < \tau < 1$, let
$\{x_1, x_2, \ldots, x_N\}$ → positions of walkers

$$P_{\rm joint}(\{x_i\}|\tau) \propto \prod_{i=1}^{N} x_i^2 \prod_{j<k}(x_j^2 - x_k^2)^2 \exp\left[-\frac{1}{2\tau(1-\tau)}\sum_i x_i^2\right]$$

(Schehr, S.M., Comtet, Randon-Furling, PRL, 101, 150601 (2008))

- $x_i^2 = \lambda_i$ → eigenvalues of the Wishart matrix $W = X^\dagger X$

  with $\beta = 2$ and $M - N = 1/2$



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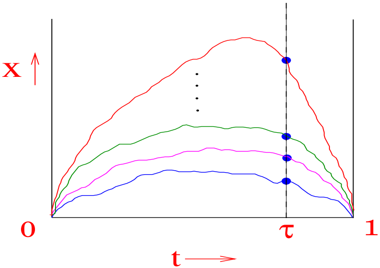
Top curve at fixed time: Tracy-Widom (GUE)



topmost curve at fixed time τ: $x_N^2(\tau) \to$ largest eigenvalue of Wishart matrices

- largest eigenvalue of the Wishart GUE matrix (properly scaled for large N) is distributed via the Tracy-Widom GUE law (Johansson 2000, Johnstone, 2001)

- This shows that the top position $x_N(\tau)$ typically fluctuates for large N as

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} = 2\sqrt{N} + 2^{-2/3}\,N^{-1/6}\,\chi_2$$

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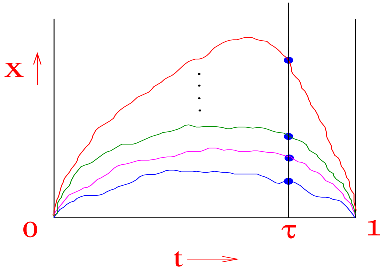

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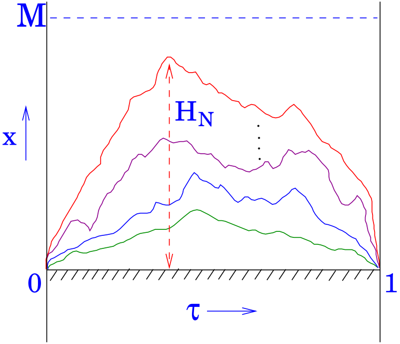
Q2: Maximal height of a watermelon with a wall



$H_N \to$ random variable

Q: What is its distribution

$\mathrm{Prob}[H_N \leq M, N] = F_N(M)$?

$N = 1$: $F_1(M) = \frac{\sqrt{2}\pi^{5/2}}{M^3} \sum_{k=1}^{\infty} k^2\, e^{-\pi^2 k^2/2M^2}$ (Chung '75, Kennedy '76)

$N = 2$, $F_2(M) \to$ complicated (Katori et. al., 2008)

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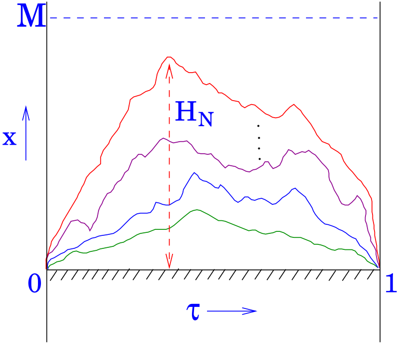

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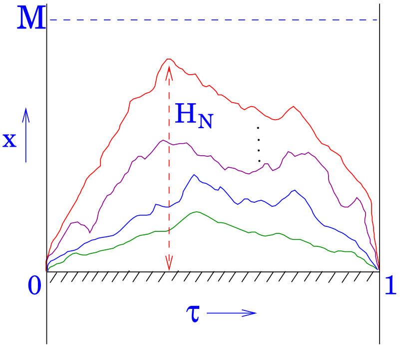

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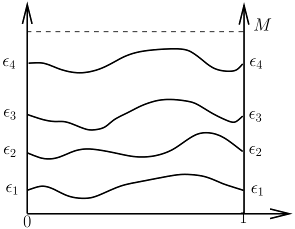
Exact result for all N via Fermionic path integral



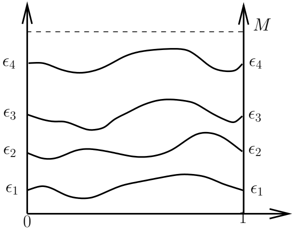
$$F_N(M) = \mathrm{Proba}[x_N(\tau) \le M,\ \forall \tau \in [0,1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

$R_M(1) \equiv$ proba. that N walkers return to their initial positions at $\tau = 1$

- $R_M(1) = \langle \vec{\epsilon} | e^{-\hat{H}} | \vec{\epsilon} \rangle = = \sum_E |\psi_E(\vec{\epsilon})|^2 \, e^{-E}$
- $\hat{H} \equiv \sum_i \left[-\frac{1}{2}\partial_{x_i}^2 + V(x_i) \right]$
- potential $V(x) = 0$ for $0 < x < M$
  $= \infty$ for $x = 0,\ M$ (Absorbing b.c.)
- $\psi_E(\vec{\epsilon}) \equiv \det\left[\sin\left(n_i \pi \epsilon_j / M\right)\right] \to$ Slater determinant ($N \times N$)
- Energy $E = \frac{\pi^2}{2M^2}\left(n_1^2 + n_2^2 + \ldots + n_N^2\right)$

Exact result for all N via Fermionic path integral



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Exact result for all N via Fermionic path integral

Using Fermionic path-integral techniques we derived the full Prob. Dist. of H_N for all N exactly

Cumul. distr: $F_N(M) = \mathrm{Prob}[H_N \leq M]$

$$F_N(M) = \frac{B_N}{M^{2N^2+N}} \sum_{n_i=1,2\ldots} \prod_{i=1}^{N} n_i^2 \prod_{j<k} (n_j^2 - n_k^2)^2 \exp\left[-\frac{\pi^2}{2M^2} \sum_i n_i^2\right]$$

where

$$B_N = \frac{\pi^{2N^2+N}\, 2^{N/2-N^2}}{\prod_{j=0}^{N-1} \Gamma(j+2)\Gamma(j+3/2)}$$

[Schehr, S.M., Comtet, Randon-Furling, PRL, 101, 150601 (2008)]

Q: Asymptotics of $F_N(M)$ as a function of M for large but fixed N?

$\longrightarrow$ Yang-Mills gauge theory on a 2-d sphere

Exploiting this correspondence to YM_2 we find:

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Using Fermionic path-integral techniques we derived the full Prob. Dist. of H_N for all N exactly

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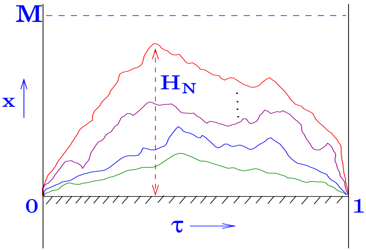
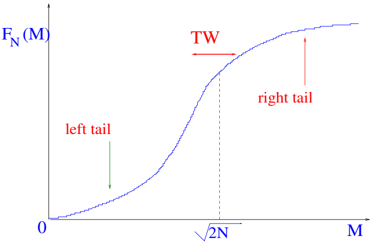
Asymptotic large N results:





Cum. distr. $F_N(M) = \mathrm{Prob.}[H_N \leq M]$ behaves, for large N as:

$$\sim \quad \exp\left[-N^2\,\Phi_-\left(\frac{M}{\sqrt{2N}}\right)\right] \qquad \text{for} \quad \sqrt{2N} - M \sim O(\sqrt{N})$$

$$\sim \quad \mathcal{F}_1\left[2^{11/6}\,N^{1/6}\,(M - \sqrt{2N})\right] \qquad \text{for} \quad |M - \sqrt{2N}| \sim O(N^{-1/6})$$

$$\sim \quad 1 - B\exp\left[-\beta N\,\Phi_+\left(\frac{M}{\sqrt{2N}}\right)\right] \qquad \text{for} \quad M - \sqrt{2N} \sim O(\sqrt{N})$$

Asymptotic large N results:

- where $\mathcal{F}_1(x) \to$ Tracy-Widom GOE
- $\phi_\pm(x) \to$ left and right rate functions $\Longrightarrow$ explicitly computable

(Schehr, S.M., Comtet, Forrester, 2011/2012)

Right rate function:

$$\phi_+(x) = 4\,x\,\sqrt{x^2-1} - 2\ln\left[2x\left(\sqrt{x^2-1}+x\right)-1\right]$$

Left rate function:

$\phi_-(x) \to$ can be expressed in terms of elliptic functions

- In particular,

$$\begin{aligned}\phi_+(x) &\simeq \frac{2^{9/2}}{3}(x-1)^{3/2} \quad \text{as} \quad x \to 1^+ \\ \phi_-(x) &\simeq \frac{16}{3}(1-x)^3 \quad \text{as} \quad x \to 1^{-1}\end{aligned}$$

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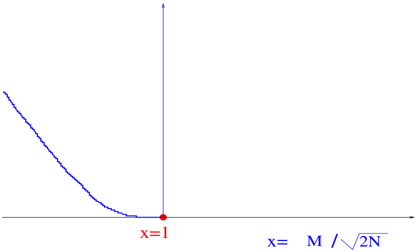
3-rd order phase transition



$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N \left(M = \sqrt{2N}\, x \right) = \begin{cases} \phi_-(x)\ , & x < 1 \\ 0\ , & x > 1\ . \end{cases}$$

Since, $\phi_-(x) \sim (1-x)^3 \Rightarrow$ 3-rd order phase transition

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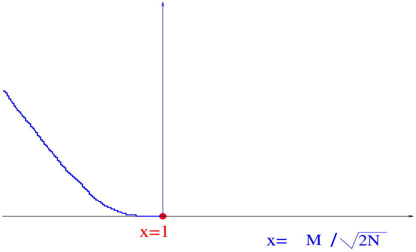

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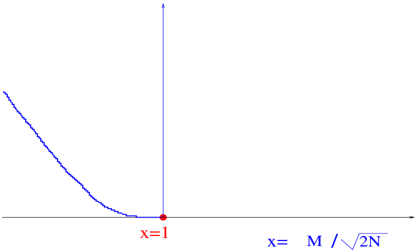

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III : Yang-Mills gauge theory in 2-d

Partition function of Yang-Mills theory in 2d

- Consider a 2-d manifold $\mathcal{M}$. At each point x: a pair of $N \times N$ matrix

$$A_\mu(x) \ (\mu = 1, 2) \rightarrow \text{gauge field}$$

Partition function: $\mathcal{Z}_\mathcal{M} = \int [\mathcal{D}A_\mu] e^{-\frac{1}{4\lambda^2} \int \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] d^2x}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \rightarrow \text{field strength}$$

$$\lambda \rightarrow \text{coupling strength}$$

- Under a local gauge transformation:

$$A_\mu \rightarrow S^{-1}(x) A_\mu S(x) - i\, S^{-1}(x) \partial_\mu S(x)$$

where $S(x) \rightarrow N \times N$ matrix that depends on the underlying gauge group G

Field strengths transform as $F_{\mu\nu} \rightarrow S^{-1}(x) F_{\mu\nu}(x) S(x)$ that keeps the action gauge invariant.

Ex: $G \equiv U(1)$: electrodynamics
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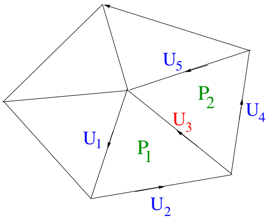
Lattice Regularization:

Consider, for instance, the $U(N)$ gauge theory

Regularization on the lattice:

$$\mathcal{Z}_{\mathcal{M}} = \int \prod_{L} dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

$$U_P = \prod_{L \in \text{plaquette}} U_L$$



$Z_P \rightarrow$ plaquette partition function

(Wilson, '74, Migdal, '75)

Heat-kernel action

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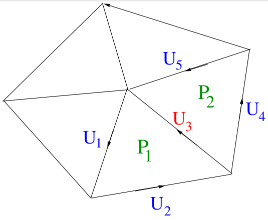

Wilson'74

- A common choice : Wilson's action

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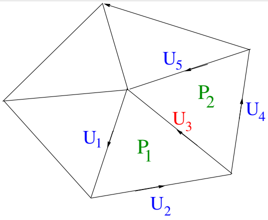

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- fixed point action : invariance under decimation ⇒ Migdal's recursion relation

$$\int dU_3\, Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^\dagger) = Z_{P_1+P_2}(U_1 U_2 U_4 U_5)$$

$$Z_P(U_P) = \sum_R d_R \chi_R(U_P) \exp\left[-\frac{A_P}{2N} C_2(R)\right]$$

Migdal'75, Rusakov'90

Partition function of Yang-Mills theory on the 2d-sphere

- Partition funct° on $\mathcal{M}$, of genus g, computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^{2-2g} \exp\left[-\frac{A}{2N} C_2(R)\right]$$

Partition function of Yang-Mills theory on the 2d-sphere

- Partition funct^{o} on the sphere computed with the heat-kernel action

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- Irreducible representations R of G are labelled by the lengths of the Young diagrams:
  - If $G = \mathrm{U}(N)$

$$\mathcal{Z}_{\mathcal{M}} = c_N\, e^{-A\frac{N^2-1}{24}} \sum_{n_1,\ldots,n_N=0}^{\infty} \prod_{i<j} (n_i - n_j)^2 e^{-\frac{A}{2N}\sum_{j=1}^{N} n_j^2}$$

  - If $G = \mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \hat{c}_N\, e^{A\,(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N}\sum_{j=1}^{N} n_j^2}$$

Correspondence between YM$_2$ on the sphere and watermelons

- Partition function of YM$_2$ on the sphere with gauge group $\mathrm{Sp}(2N)$

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- Cumulative distribution of the maximal height of watermelons with a wall

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$$\propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N)\right)$$

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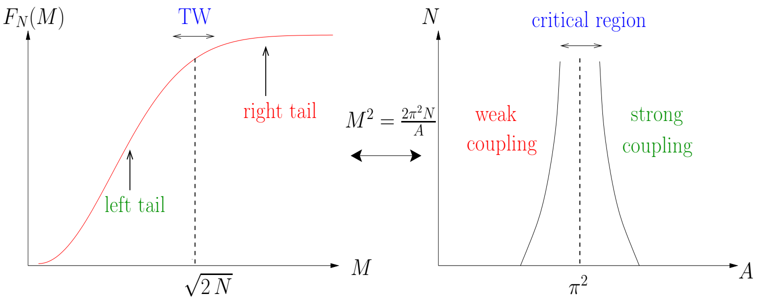
Weak-strong coupling transition (3-rd order) in YM_2, Douglas-Kazakov '93



Critical point $A = A_c = \pi^2$ corresponds (using $A = \frac{2\pi^2 N}{M^2}$):

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$A > A_c$ (Strong Coupling) $\longleftrightarrow$ $M < M_c = \sqrt{2N}$ (left tail of H_N)

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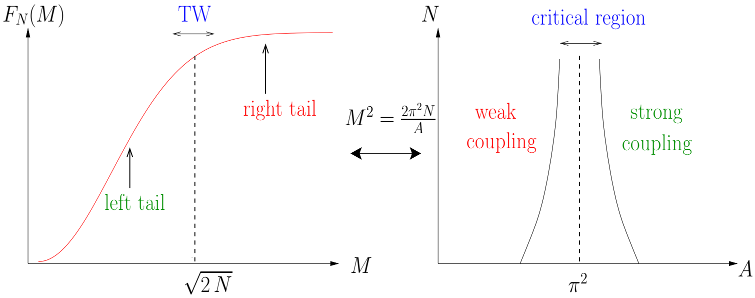

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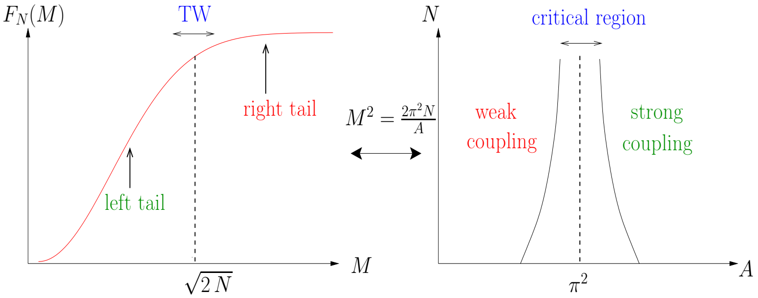

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$A < A_c$ (Weak Coupling) $\longleftrightarrow$ $M > M_c = \sqrt{2N}$ (right tail of H_N)

Large N limit of YM_2 and consequences for $F_N(M)$

- In the critical regime, "double-scaling limit", the method of orthogonal polynomials (Gross-Matytsin '94, Crescimanno-Naculich-Schnitzer '96) shows

$$\frac{d^2}{dt^2}\log F_N\left(\sqrt{2N}(1+t/(2^{7/3}N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t)-q'(t)\right)$$
$$q''(t) = 2q^3(t)+t\,q(t)\ ,\ q(t)\sim \mathrm{Ai}(t)\ ,\ t\to\infty$$

$$F_N(M) \to \mathcal{F}_1\left(2^{11/6}N^{1/6}\left|M-\sqrt{2N}\right|\right)$$
$$\mathcal{F}_1(t) = \exp\left(-\frac{1}{2}\int_t^\infty\left((s-t)\,q^2(s)+q(s)\right)\,ds\right)$$
$$\equiv \text{Tracy-Widom distribution for } \beta=1$$

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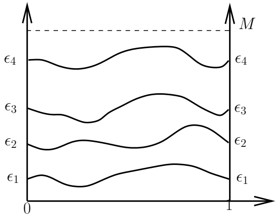
- double scaling regime $[A\sim A_c]$ $\longleftrightarrow$ Tracy-Widom $[M\sim\sqrt{2N}]$

Forrester, S. M., Schehr, '11

Absorbing boundary condition $\to SP(2N)$

- Ratio of reunion probabilities for N vicious walkers on the segment $[0, M]$ with absorbing boundary conditions



$$F_N(M) = \mathrm{Proba}[x_N(\tau) \leq M, \; \forall \tau \in [0, 1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

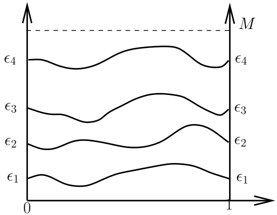
$R_M(1) \equiv$ proba. that N walkers return to their initial positions at $\tau = 1$

Related to YM_2 on the sphere with gauge group $\mathrm{Sp}(2N)$

$$F_N(M) \propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N)\right)$$

limiting form of $F_N(M)$: $\mathcal{F}_1 \to$ Tracy-Widom (GOE)

- Ratio of reunion probabilities for N vicious walkers on the segment $[0, M]$ with periodic boundary conditions



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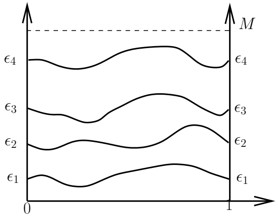

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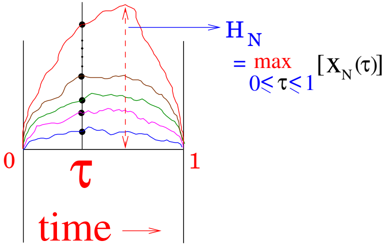
limiting form of $F_N(M)$: $\frac{\mathcal{F}_2}{\mathcal{F}_1}$

Summary



$x_i(\tau) \to$ trajectory of the i-th walker
$x_N(\tau) \to$ trajectory of the top path
$x_N(\tau)$ (centered and scaled)
$\to$ Airy$_2$ process minus a parabola

Prähofer & Spohn, '00

- At fixed time τ, the marginal $x_N(\tau)$ (centered and scaled)
  $\to$ Tracy-Widom $\beta = 2$

- However, the maximal height $\boxed{H_N = \max_{0\le\tau\le 1}[x_N(\tau)]}$ (centered and scaled)

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  an indirect proof via PNG growth model $\to$ Johansson (2003)

- beautiful connection to YM_2 and the 3-rd order phase transition

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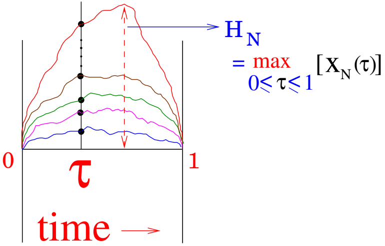

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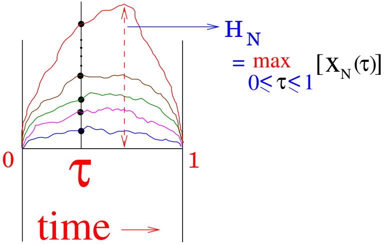

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Conclusion

WISHART RANDOM MATRICES

Wishart 1928, Tracy–Widom 1993, Johansson 2000 ....

VICIOUS BROWNIAN WALKERS

de Gennes 1968, Fisher 1984, ...

2–d YANG–MILLS THEORY ON THE SPHERE

LARGE N PHASE TRANSITION (3rd order)

LATTICE (Wilson Action): Gross and Witten 1980, Wadia 1980....

CONTINUUM : Migdal 1975 , Rusakov 1990, Douglas and Kazakov 1993, Gross and Matytsin 1994....

Collaborators and References

Collaborators:

- O. Bohigas, A. Comtet, G. Schehr, P. Vivo (LPTMS, Orsay, France)
- P. J. Forrester (Univ. of Melbourne, Australia)
- C. Nadal (Oxford University, UK)
- J. Randon-Furling (Univ. Paris-1, France)
- M. Vergassola (Inst. Pasteur, Paris, France)

References:

- P. Vivo, S. M., O. Bohigas, J. Phys. A: Math. Theo. **40**, 4317 (2007).
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- G. Schehr, S. M., A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008).
- C. Nadal, S. M., Phys. Rev. E **79**, 061117 (2009).
- P. J. Forrester, S. M., G. Schehr, Nucl. Phys. B **844**, 500 (2011).
- G. Schehr, S. M., A. Comtet, P. J. Forrester, J. Stat. Phys. **150**, 491 (2013).

Open Questions and related issues:

- boundary conditions $\Longleftrightarrow$ gauge groups
  deeper understanding needed

- Other interesting observables:

- Joint distribution of the maximal height $H_N = \max_{0 \le \tau \le 1}[x_N(\tau)]$ and the time τ_M at which it occurs: $P_N(H_N = M, \tau_M)$

  $\Longrightarrow$ Interesting relation to KPZ interfaces and $(1+1)$-d directed polymers

Rambeau & Schehr '11, Flores et. al. '12, Schehr '12, Quastel & Remenik, '12, Baik, Liechty, Schehr, '12

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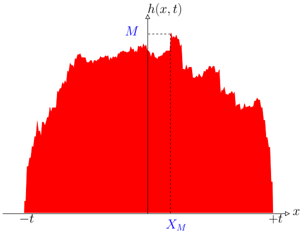
Consequences for curved stochastic growth

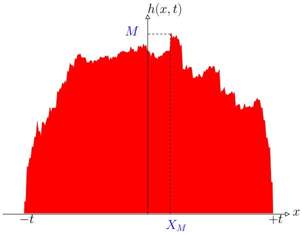


- Distribution of the height field $h(0,t)$ (Prähofer & Spohn, '00)

$$\lim_{t\to\infty} P\left(\frac{h(0,t)-2t}{t^{1/3}} \leq s\right) = \mathcal{F}_2(s)$$

$\mathcal{F}_2(s) \equiv$ Tracy – Widom distribution for $\beta = 2$

Consequences for curved stochastic growth



- Maximum $M \equiv \max_{-t \leq x \leq t} h(x,t)$ (Forrester, S.M. and Schehr, NPB '11)

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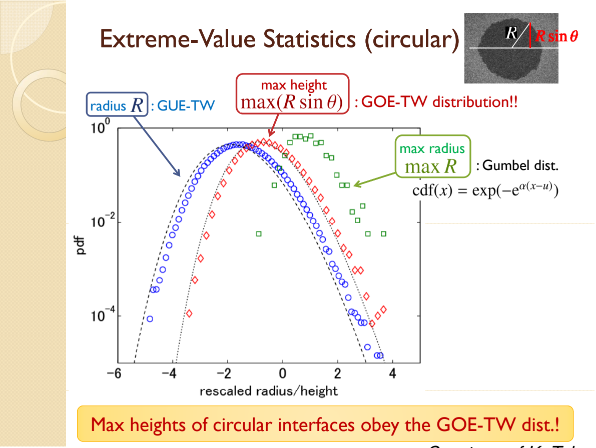
see also

- Krug *et al.* '92, Johansson '03 (indirect proof),
- G. M. Flores, J. Quastel, D. Remenik, arXiv:1106.2716

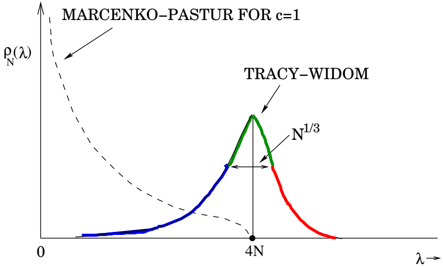
Experiments on nematic liquid crystals

K. A. Takeuchi, M. Sano, Phys. Rev. Lett. **104**, 230601 (2010)



Courtesy of K. Takeuchi

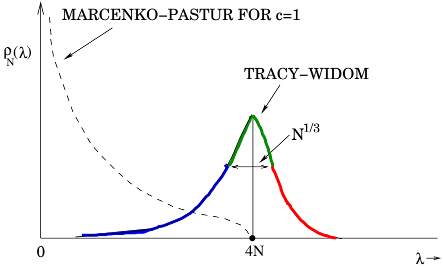
Probability of Atypically Large Deviations of $\lambda_{\max}$:



- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \leq t, N] \to F_\beta \left[\frac{t-4N}{2^{4/3}N^{1/3}}\right]$ describes the prob. of typical (small) fluctuations of $\sim O(N^{1/3})$ around the mean $4N$, i.e., when $|\lambda_{\max} - 4N| \sim N^{1/3}$

- Q: the prob. of large (atypical) fluctuations (red and blue)?

$$|\lambda_{\max} - 4N| \sim O(N)$$



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Exact Left and Right Large Deviation Functions

- For large deviation: $t - 4N \sim O(N)$

$$P(\lambda_{\max} = t, N) \approx \begin{cases} \exp\left\{-\beta N^2\, \Psi_-\left(\frac{4N-t}{N}\right)\right\} & \text{for } t << 4N \\ \exp\left\{-\beta N\, \Psi_+\left(\frac{t-4N}{N}\right)\right\} & \text{for } t >> 4N \end{cases}$$

- $\Psi_-(x)$ and $\Psi_+(x)$ $\rightarrow$ computed exactly

(Vivo, S.M. and Bohigas 2007, S.M. and Vergassola 2009)

$$\Psi_-(x) \rightarrow \frac{x^3}{384} \text{ (as } x \rightarrow 0)$$
$$\Psi_+(x) \rightarrow \frac{x^{3/2}}{6} \text{ (as } x \rightarrow 0)$$

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- Left large deviation function:

$$\Psi_-(x) = \ln\left[\frac{2}{\sqrt{4-x}}\right] - \frac{x}{8} - \frac{x^2}{64}; \quad x \geq 0$$

(Vivo, S.M., and Bohigas, 2007)

- Right large deviation function:

$$\Psi_+(x) = \frac{1}{2}\sqrt{x(x+4)} + \ln\left[\frac{x+2-\sqrt{x(x+4)}}{2}\right]; \quad x \geq 0$$

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$$\lim_{N\to\infty} -\frac{1}{\beta N^2} \ln\left[P\left(\lambda_{\max} \leq 4N - N\,x, N\right)\right] = \begin{cases} \Psi_-(x) \sim x^3 & \text{as } x \to 0^- \\ 0 & \text{as } x \to 0^+ \end{cases}$$

3-rd derivative $\to$ discontinuous

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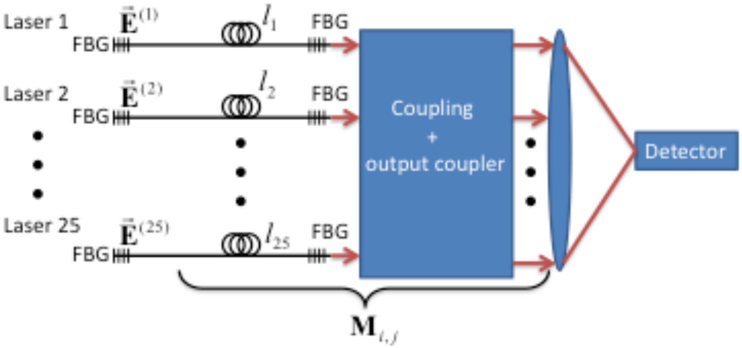
Measuring maximal eigenvalue distribution of Wi
lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Ashe
Weizmann Institute of Science, Dept. of Physics of Co
(Dated: May 30, 201

We determined the probability distribution of the combine
fiber lasers and show that it agrees well with the Tracy-W
Majumdar-Bohigas distributions of the largest eigenvalue
fitting parameters. This was achieved with 500,000 measu
from the fiber lasers, that continuously changes with var
show experimentally that for small deviations of the comb
the Tracy-Widom distribution is correct, while for large de
Vivo-Majumdar-Bohigas distributions are correct.

Experimental Verification with Coupled Lasers



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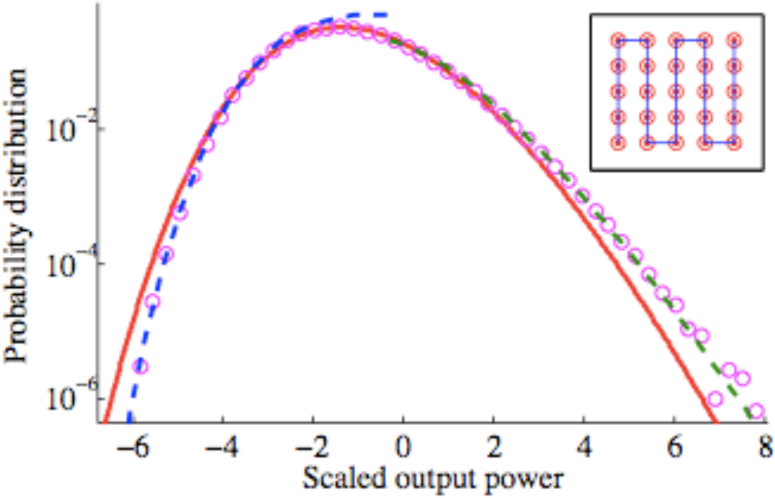

(a)

Experimental Verification with Coupled Lasers

Fridman et. al. arXiv: 1012.1282

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