Lectures Notes

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- Conformal Field Theories and Conformal Manifolds
- Trace Anomalies, 't Hooft Anomalies, Gravitational Anomalies
- Anomalies on Conformal Manifolds
- Monotonicity Theorems from old and new Points of View
- Sphere Partition Functions and $\mathcal{N} = 2$ Theories in Four Dimensions
- Three-Dimensional Theories
- Supersymmetric Indices
- Scale vs. Conformal Invariance
- S-Matrix Theory

There are certainly many typos, errors etc.

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Chapter 1
The Conformal Algebra and Conformal Manifolds

A special class of Quantum Field Theories (QFTs) are those that have no intrinsic length scale. This happens when the correlation length of the corresponding theory on the lattice diverges. In addition, such theories often arise when we take generic QFTs and scale the distances to be much larger or much smaller than the typical inverse mass scales. Of course, one may sometimes encounter gapped theories at long distances, but there are also many examples in which one finds nontrivial theories in this way.

In general, we are interested here in QFTs which are invariant under the Poincaré group of R^4. The Poincaré group consists of rotations in $SO(4)$ (generated by $M_{\mu\nu}$, with $(\mu,\nu = 1,...,4)$) and translations (generated by P_μ). If the theory has no intrinsic length scale then the Poincaré group is enhanced by adding the generator of dilations, Δ. Oftentimes, the symmetry is further enhanced to $SO(5,1)$, which includes the original Poincaré generators, the dilation Δ, and the so-called special conformal transformations K_μ.

The commutation relations are

$$
\begin{aligned}
\left[\Delta, P_\mu\right] &= P_\mu \ , \\
\left[\Delta, K_\mu\right] &= -K_\mu \ , \\
\left[K_\mu, P_\nu\right] &= 2\left(\delta_{\mu\nu}\Delta - iM_{\mu\nu}\right) \ , \\
\left[M_{\mu\nu}, P_\rho\right] &= i\left(\delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu\right) \ , \\
\left[M_{\mu\nu}, K_\rho\right] &= i\left(\delta_{\mu\rho}K_\nu - \delta_{\nu\rho}K_\mu\right) \ , \\
\left[M_{\mu\nu}, M_{\rho\sigma}\right] &= i\left(\delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\rho} - \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\sigma}M_{\nu\rho}\right) \ .
\end{aligned}
$$

They can be realized by the differential operators acting on R^4:

$$
\begin{aligned}
M_{\mu\nu} &= -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \ , \\
P_\mu &= -i\partial_\mu \ , \\
K_\mu &= i\left(2x_\mu x.\partial - x^2\partial_\mu\right) \ , \\
\Delta &= x.\partial \ .
\end{aligned}
$$

A primary operator, $\Phi(x)$, is, by definition, an operator that is annihilated by K_μ when placed at the origin:

$$[K_\mu, \Phi(0)] = 0 \ . \tag{1}$$

The origin of R^4 is fixed by rotations and dilations. Therefore we can characterise $\Phi(0)$ by its quantum numbers under rotations and dilations. In $d = 4$ the group of rotations, $SO(4)$, is just $SU(2) \times SU(2)$ and hence a primary operator is labeled by $(j_1, j_2; \Delta)$.[1] We will be only interested in unitary theories, where the allowed representations of the conformal algebra do not have negative-norm states.

It is sometimes the case that the conformal field theory has primary operators of dimension 4,

$$[\Delta, O_I(x)] = i(x.\partial + 4)O_I(x) \ .$$

If we add such operators to the action with couplings λ^I then we get

$$S \to S + \sum_I \lambda^I \int d^4x O_I(x) \ . \tag{2}$$

A simple example is the free conformal field theory in $d = 4$ to which we can add a quartic interaction. The coupling λ^I is dimensionless but in general there may be a nontrivial beta function

$$\beta^I \equiv \frac{d\lambda^I}{d\log\mu} = (\beta^{(1)})^I_{JK}\lambda^J\lambda^K + \cdots \ . \tag{3}$$

Therefore, conformal symmetry is broken at second order in λ. (If we add to the action an operator of $\Delta \neq 4$ then conformal symmetry is already broken at first order in the coupling constant.)

We can compute the $(\beta^{(1)})^I_{JK}$ by imagining a correlation function $\langle\cdots\rangle$ which we expand as a function of λ^I. We get

$$\langle\cdots\rangle_{\lambda=0} - \lambda^I \int d^4x \langle O_I(x)\cdots\rangle_{\lambda=0} + \frac{1}{2}\lambda^J\lambda^K \int d^4y \int d^4z \langle O_J(y)O_K(z)\cdots\rangle_{\lambda=0} + \ldots$$

There is a logarithmic divergence from $y \to z$ due to the OPE

$$O_J(y)O_K(z) \sim C^I_{JK}\frac{O_I(y)}{(y-z)^d}$$

We now imagine integrating a-la Wilson from some UV cutoff μ_{UV} to some lower cutoff μ so we pick $Vol(S^3)\log(\mu_{UV}/\mu)$. (Recall $Vol(S^3) = 2\pi^2$.) Therefore, we see that an effective scale dependent coupling is generated, $\lambda^I(\mu)$, such that

$$\lambda^I(\mu) = \lambda^I(\mu_{UV}) - \pi^2\lambda^J(\mu_{UV})\lambda^K(\mu_{UV})C^I_{JK}\log\left(\frac{\mu_{UV}}{\mu}\right) + \cdots \ .$$

[1] If there is a global symmetry $\mathscr{G}$, then the primary operators would furnish some representations of $\mathscr{G}$.

Therefore,

$$\frac{d}{d\log\mu}\lambda^I = \pi^2 C^I_{JK}\lambda^J\lambda^K + \cdots \ .$$

This allows to identify the "one-loop" beta functions with the corresponding OPE coefficients

$$(\beta^{(1)})^I_{JK} = C^I_{JK} \ .$$

Exercise 1 (easy): Check that this gives the correct one-loop beta function for the ϕ^4 theory

Exercise 2 (hard): Extend this theory to order λ^3

Under some special circumstances it may happen that $\beta^I = 0$ as a function of λ^I (to all orders). Then we say that the deformation (2) is exactly marginal. We therefore have a manifold of conformal field theories, $\mathscr{M}$, with coordinates $\{\lambda^I\}$. This manifold has a natural Riemannian structure given by the Zamolodchikov metric

$$\langle O_I(0) O_J(\infty)\rangle_{\{\lambda^I\}} = g_{IJ}(\lambda^I) \ . \tag{4}$$

A primary operator $\Phi(\infty)$ is defined by the limit $\Phi(\infty) = \lim_{y\to\infty} y^{2\Delta}\Phi(y)$ with Δ being the dimension of Φ. This endows the conformal manifold $\mathscr{M}$ with a Riemannian structure. We cannot put globally $g_{IJ}(\lambda^I) = \delta_{IJ}$ but we can do it at some given point, p, analogously to choosing Riemann normal coordinates in some patch that includes $p \in \mathscr{M}$.

In Quantum Field Theory, the usual freedom in choosing a metric is reflected by a redefinition of the coupling constants. We can replace (2) with

$$S \to S + \sum_I F^I(\lambda^J) \int d^4x O_I(x) \ . \tag{5}$$

This only differs from (2) by what one means by the coupling constant. The metric is now given by

$$\frac{\partial F^J}{\partial \lambda^I}\frac{\partial F^K}{\partial \lambda^J} g_{JK} \ ,$$

which is the usual transformation rule for the metric. The Riemann tensor carries the information about the obstruction to putting the metric to δ_{IJ} in some patch. Let us take some Riemann normal coordinates centred at $\lambda^I = 0$. Then the Riemann tensor is given by

$$R_{ijkl} = \int_A d^4z \langle O_i(0) O_k(z) O_l(1) O_j(\infty)\rangle - k \leftrightarrow l \ ,$$

with the region A defined as

$$A \equiv \{z \in \mathbb{R}^4 |\ |z| < |1-z|, |z| < 1\} \ .$$

Exercise 3: Show that this definition respects the anti-symmetry of the Riemann tensor in the $i-j$ indices and it satisfies the interchange symmetry $R_{ijkl} = R_{klij}$.

One situation in which exactly marginal operators are common is in Super-Conformal Field Theories (SCFTs). The conformal algebra is enlarged by adding $\mathcal{N}$ Poincaré supercharges $Q^i_\alpha, \bar{Q}_{i\dot{\alpha}}$ and $\mathcal{N}$ superconformal supercharges $S_i^\alpha, \bar{S}^{i\dot{\alpha}}$ ($i = 1,..,\mathcal{N}$). In addition, we must add the R-symmetry group, $U(\mathcal{N})$, whose generators are R^i_j. This furnishes the superalgebra $SU(2,2|\mathcal{N})$.

We do not list all the commutation relations. They can be found in [5]. All we need to know for our purposes is summarised below.

Our main interest in this note lies in $\mathcal{N} = 2$ theories. The maximally supersymmetric theory with $\mathcal{N} = 4$ would be a special case. The R-symmetry group in $\mathcal{N} = 2$ theories is $SU(2)_R \times U(1)_R$. We denote the $U(1)_R$ charge by r.

- It is consistent to impose at the origin, in addition to (1),
$$[S_i^\alpha, \Phi(0)] = [\bar{S}^{i\dot{\alpha}}, \Phi(0)] = 0 \ .$$
(The quantum numbers of $\Phi(0)$ are omitted.) Such operators are called superconformal primaries. In every unitary representation the operators with the lowest eigenvalues of Δ are superconformal primaries.
- If one further imposes
$$[\bar{Q}_{i\dot{\alpha}}, \Phi(0)] = 0 \ , \tag{6}$$
one obtains a short representation (such representations may or may not exist in a given model). The operator $\Phi(0)$ satisfying (6) is called a chiral primary.[2] Chiral primary operators are necessarily $SU(2)_R$ singlets and they obey a relationship between their $U(1)_R$ charge and their scaling dimension
$$\Delta = r \ .$$
- Marginal operators that preserve $\mathcal{N} = 2$ supersymmetry are necessarily the descendants of chiral primary operators with $\Delta = r = 2$. We can upgrade the formula (2) to a superspace formula
$$S \to S + \lambda^I \int d^4x \, d^4\theta \, \Phi_I(x,\theta) + \bar{\lambda}^{\bar{I}} \int d^4x \, d^4\bar{\theta} \, \bar{\Phi}_{\bar{I}}(x,\bar{\theta}) \ . \tag{7}$$
which shows that $\mathcal{N} = 2$ supersymmetry is indeed preserved. We denote the dimension 4 descendant of Φ_I by O_I. Therefore, (7) is just
$$S \to S + \lambda^I \int d^4x \ O_I(x) + \bar{\lambda}^{\bar{I}} \int d^4x \ \bar{O}_{\bar{I}}(x) \ . \tag{8}$$
The Zamolodchikov metric is defined by the two-point function $\langle O_I(\infty)\bar{O}_{\bar{J}}(0)\rangle$. (This is proportional to $\langle \Phi_I(\infty)\bar{\Phi}_{\bar{J}}(0)\rangle$.)

One can actually prove that for the deformations (7) the beta function $\beta^I = 0$ identically. The argument is along the lines of [11]. There is a scheme in which the

[2] We henceforth assume chiral primary operators carry no spin, see [4] for a discussion.

superpotential is not renormalized. Then if the beta function is nonzero it has to be reflected by a D-term in the action $\int d^4x\, d^8\theta\, \mathscr{O}$ with $\mathscr{O}$ some real primary operator. But since the λ^I are classically dimensionless, $\Delta(\mathscr{O}) = 0$ in the original fixed point. Therefore, $\mathscr{O}$ has to be the unit operator and the deformation $\int d^4x\, d^8\theta\, \mathscr{O}$ is therefore trivial. This proves that $\beta^I = 0$.

The $\{\lambda^I, \bar{\lambda}^{\bar{I}}\}$ are therefore coordinates on the manifold $\mathscr{M}$ of $\mathscr{N} = 2$ SCFTs. It readily follows from (7) that the manifold $\mathscr{M}$ is Hermitian in this case. Indeed, the allowed changes of variables in (7) are holomorphic $\lambda^I \to F^I(\lambda^J)$ and therefore there is a complex structure on $\mathscr{M}$. The only non-vanishing components of the metric are the mixed components, given by $\langle O_I(\infty)\bar{O}_{\bar{J}}(0)\rangle$, and therefore the manifold $\mathscr{M}$ is Hermitian. (The other components of the metric vanish by a SUSY Ward identity.)

Soon we will argue that $\mathscr{M}$ is a Kähler manifold, i.e. the Zamolodchikov metric (4) satisfies

$$g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K(\lambda^I, \bar{\lambda}^{\bar{I}}) \ . \tag{9}$$

Then we will argue that the Kähler potential can be extracted from the S^4 partition function and we will use supersymmetric localization to compute it in some simple $\mathscr{N} = 2$ SCFTs. Large parts of this discussion follows [10].

In order to understand these ideas we have to first go through the theory of quantum anomalies.

Chapter 2
Anomalies in Two-Dimensional Models

We consider Euclidean two-dimensional theories that enjoy the isometry group of $\mathbb{R}^2$. Namely, the theories are invariant under translations and rotations in the two space directions. If the theory is local it has an energy momentum tensor operator $T_{\mu\nu}$ which is symmetric and conserved $T_{\mu\nu} = T_{\nu\mu}$, $\partial^\mu T_{\mu\nu} = 0$. These equations should be interpreted as operator equations, namely they must hold in all correlation functions except, perhaps, at coincident points. Here we will study theories where, when possible, these equations hold in fact also at coincident points. In other words, we study theories where there is no local gravitational anomaly. See [17] for some basic facts about theories violating this assumption.

We can study two-point correlation functions of the energy-momentum operator. This correlation function is highly constrained by symmetry and conservation. It takes the following most general form in momentum space:

$$\begin{aligned}\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle &= \frac{1}{2}\left((q_\mu q_\rho - q^2\eta_{\mu\rho})(q_\nu q_\sigma - q^2\eta_{\nu\sigma}) + \rho \leftrightarrow \sigma\right) f(q^2) \\ &+ (q_\mu q_\nu - q^2\eta_{\mu\nu})(q_\rho q_\sigma - q^2\eta_{\rho\sigma}) g(q^2)\ .\end{aligned} \tag{10}$$

The most general two-point function is therefore fixed by two unknown functions of the momentum squared. (Note that above we have stripped the trivial delta function enforcing momentum conservation.)

The form (10) holds in any number of dimensions. In two dimensions the two functions $f(q^2)$ and $g(q^2)$ are not independent.

Exercise 4: Show that the two kinematical structures in (10) **are not independent in two dimensions.**

Now let us make a further assumption, that the theory is scale invariant (but not necessarily conformal invariant). This allows us to fix the two functions f, g up to a constant[3]

$$f(q^2) = \frac{b}{q^2}\ , \qquad g(q^2) = \frac{d}{q^2}\ . \tag{11}$$

[3] A logarithm is disallowed since it violates scale invariance. (The rescaling of the momentum would not produce a local term.)

We can now calculate the two-point function $\langle T^\mu_\mu(q)T^\mu_\mu(-q)\rangle$

$$\langle T^\mu_\mu(q)T^\mu_\mu(-q)\rangle = (b+d)q^2 \ . \tag{12}$$

Transforming back to position space this means that $\langle T^\mu_\mu(x)T^\mu_\mu(0)\rangle \sim (b+d)\Box\delta^{(2)}(x)$. In particular, at separated points the correlation function vanishes:

$$\langle T^\mu_\mu(x)T^\nu_\nu(0)\rangle_{x\neq 0} = 0 \ . \tag{13}$$

In unitary theories, this means that the trace itself is a vanishing operator $T^\mu_\mu = 0$ (since it creates nothing from the vacuum, it must be a trivial operator). This is called the Reeh-Schlieder theorem [18].

The operator equation

$$T^\mu_\mu = 0 \tag{14}$$

is precisely the condition for having the full conformal symmetry of $\mathbb{R}^2$, $SO(3,1)$, present.

In fact, in two-dimensions, (14) is sufficient to guarantee an infinite symmetry group (the Virasoro algebra). This is because (in complex coordinates) T_{zz} obeys $\bar\partial T_{zz} = 0$ and so it can be multiplied by an arbitrary holomorphic function and still remain conserved. We will elaborate on that a little more later.

The equation (14) is satisfied in all correlation functions at separated points, but it may fail at coincident points. We have already seen this phenomenon in (12). Such contact terms signal an anomaly of $SO(3,1)$. Other than this potential anomaly, (14) means that $SO(3,1)$ is a perfectly good symmetry of the theory. Therefore, we see that scale invariant theories are necessarily conformal invariant in two dimensions [19].[4]

More generally, we find

$$\langle T^\rho_\rho(q)T_{\mu\nu}(-q)\rangle = -(b+d)(q_\mu q_\nu - q^2\eta_{\mu\nu}) \ , \tag{15}$$

which is again a polynomial in momentum and hence a contact term in position space, consistently with (14). One may wonder at this point whether there exist Quantum Field Theories for which these contact terms are absent because $b+d=0$. Consider the correlation function $\langle T_{11}T_{11}\rangle$. This has support at separated points. It is proportional to $b+d$.

Exercise 5: In light of exercise 4, is this a coincidence?

Therefore, in unitary theories, from reflection positivity (i.e. Reeh-Schlieder theorem) it follows that

$$b+d > 0 \ . \tag{16}$$

So far we have seen that scale invariant theories in fact enjoy $SO(3,1)$ symmetry, but the symmetry is afflicted with various contact terms such as (15). To see

[4] Some technical assumptions implicit in the argument above are spelled out in [19], where the argument was first developed.

clearly the physical meaning of this anomaly, we can couple the theory to some ambient curved space (there is no dynamics associated to the curved space, it is just a background field). This is done to linear order via $\sim \int d^2xT^{\mu\nu}h_{\mu\nu}$, where $h_{\mu\nu}$ is the linearized metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Hence in the presence of a background metric that deviates only slightly from flat space

$$\langle T^{\mu}_{\mu}(0)\rangle_{g_{\mu\nu}} \sim \int d^2x \langle T^{\mu}_{\mu}(0)T^{\rho\sigma}(x)\rangle h_{\rho\sigma}(x) \sim (b+d)\int d^2x \left(\partial^{\rho}\partial^{\sigma}\delta^2(x) - \eta^{\rho\sigma}\Box\delta^2(x)\right) h_{\rho\sigma}$$
$$\sim (b+d)(\partial^{\rho}\partial^{\sigma} - \eta^{\rho\sigma}\Box)h_{\rho\sigma} \; . \tag{17}$$

The final object $(\partial^{\rho}\partial^{\sigma} - \eta^{\rho\sigma}\Box)h_{\rho\sigma}$ is identified with the linearized Ricci scalar. In principle, if we had analyzed three-point functions of the energy-momentum tensor and so forth, we would have eventually constructed the entire series expansion of the Ricci scalar. Therefore, the expectation value of the trace of the energy-momentum tensor is proportional to the Ricci scalar of the ambient space. This is the famous two-dimensional trace anomaly. It is conventional to denote the anomaly by c (and not by $b+d$ as we have done so far). The usual normalization is

$$T = -\frac{c}{24\pi}R \; . \tag{18}$$

c is also referred to as the "central charge" but we will not emphasize this representation-theoretic interpretation here. Our argument (16) translates to $c > 0$.

There is a very useful consequence of (49). Consider the conservation equation in curved space $\nabla^{\mu}T_{\mu\nu} = 0$. If we switch to local complex coordinates with a Hermitian metric

$$ds^2 = e^{\varphi}dzd\bar{z} \; ,$$

then the conservation equation takes the form

$$\partial_{\bar{z}}T_{zz} + e^{\varphi}\partial_z(e^{-\varphi}T_{z\bar{z}}) = 0 \; .$$

But in two dimensions we have the relation (49) so we can substitute this into the second term of the equation above. It is useful to remember that $R = -4e^{-\varphi}\partial\bar{\partial}\varphi$.

This implies that we can definite a holomorphic energy-momentum tensor

$$T'_{zz} = T_{zz} + \alpha c\left(-(\partial\varphi)^2 + 2\partial^2\varphi\right) \; . \tag{19}$$

Exercise 6: Fix the numerical coefficient α and verify the appearance of the Schwartzian derivative below.

The advantage of T' is that it is holomorphic, but the disadvantage is that it transforms non-covariantly under holomorphic coordinate transformations $z \to f(z)$ (because φ shift in-homogeneously). T' is what is actually used in most of the literature on 2d CFTs, and the prime is usually omitted. The in-homogenous piece in the transformation rule is the so-called Schwartzian derivative

$$z \to w(z) \;, \qquad \{w,z\} = \frac{w_{zzz}}{w_z} - \frac{3}{2}\left(\frac{w_{zz}}{w_z}\right)^2 \;. \tag{20}$$

Using (18) we can present another useful interpretation of c. Consider a two-dimensional conformal field theory compactified on a two-sphere $\mathbb{S}^2$ of radius a,

$$ds^2 = \frac{4a^2}{(1+|x|^2)^2}\sum_{i=1}^{2}(dx_i)^2 \;, \qquad |x|^2 = \sum_{i=1}^{2}(x_i)^2 \;. \tag{21}$$

The Ricci scalar is $R = 2/a^2$. Because of the anomaly (49), the partition function

$$Z_{\mathbb{S}^2} = \int [d\Phi] e^{-\int_{\mathbb{S}^2}\mathscr{L}(\Phi)} \tag{22}$$

depends on a. (If the theory had been conformal without any anomalies, we would have expected the partition function to be independent of the radius of the sphere.) We find that

$$\frac{d}{d\log a}\log Z_{\mathbb{S}^2} = -\int_{\mathbb{S}^2}\sqrt{g}\langle T\rangle = \frac{c}{24\pi}\int_{\mathbb{S}^2}\sqrt{g}R = \frac{c}{24\pi}\frac{2}{a^2}Vol(\mathbb{S}^2) = \frac{c}{3} \;. \tag{23}$$

Thus the logarithmic derivative of the partition function yields the c anomaly. This particular interpretation of c will turn out to be very useful later.

The discussion in this section could have been simplified a lot by using light cone or holomorphic coordinates. But we have not done that in order to facilitate the comparison with $d > 2$. We will come back to the energy momentum tensor in $d = 2$ below, where we will use light-cone coordinates.

1 ’t Hooft Anomalies in $d = 2$

We would like to imagine some renormalization group flow and we study a $U(1)$ symmetry present along the flow, associated to a current obeying

$$p_+ j_- + p_- j_+ = 0 \;. \tag{24}$$

We have now in general three two-point correlation functions

$$\langle j_+ j_+\rangle = p_+^2\frac{a(p^2)}{p^2} \;, \qquad \langle j_+ j_-\rangle = -b(p^2) \;, \qquad \langle j_- j_-\rangle = p_-^2\frac{c(p^2)}{p^2} \;.$$

The functions a, b, c are all dimension 0. Conservation relates $a = b = c$. But again these have to interpreted just as equations for the imaginary part (i.e. separated points upon studying the conservation equations). So we are allowed to add a constant to the real part. Therefore the most general solution is

$$\langle j_+ j_+ \rangle = \frac{p_+^2}{p^2}\left(a(p^2/M^2)+k_L\right) \ ,$$

$$\langle j_+ j_- \rangle = -\left(a(p^2/M^2)+k\right) \ ,$$

$$\langle j_- j_- \rangle = \frac{p_-^2}{p^2}\left(a(p^2/M^2)+k_R\right) \ .$$

Above k_L and k_R clearly affect the correlation functions at separated points but k is just a contact term that we can set to whatever we want.

If $k_L = k_R$ then we can respect the conservation equation also at coincident points by choosing $k = k_L = k_R$. However if $k_L \neq k_R$ there is no preferred choice for k and we will just set $k = 0$. Therefore, our final result is

$$\begin{aligned}\langle j_+ j_+ \rangle &= \tfrac{p_+^2}{p^2}\left(a(p^2/M^2)+k_L\right) \ ,\\ \langle j_+ j_- \rangle &= -a(p^2/M^2) \ ,\\ \langle j_- j_- \rangle &= \tfrac{p_-^2}{p^2}\left(a(p^2/M^2)+k_R\right) \ .\end{aligned} \tag{25}$$

We would like to identity the $k_{L,R}$ as the current algebra levels in the UV so we take (without loss of generality)

$$\lim_{p^2\to\infty} a(p^2/M^2) = 0 \ .$$

Notice that in the ultraviolet, i.e. setting $a = 0$, we can flip the sign of j_- and still get the same correlation functions, which means that also the equation $p_- j_+ - p_+ j_- = 0$ is obeyed up to coincident point anomalies. Suppose in the ultraviolet we had $k_L = k_R$. Then, the equation $p_- j_+ - p_+ j_- = 0$ is obeyed including at coincident points if we re-introduce the seagull contact-term k. So the ('vector') current (j_+, j_-) is non-anomalous. But the ('axial') current $(j_+, -j_-)$ is anomalous. We have

$$\langle (p_- j_+ - p_+ j_-) j_+ \rangle = k p_+$$

and

$$\langle (p_- j_+ - p_+ j_-) j_- \rangle = -k p_-$$

Upon coupling $j_+ A_- + j_- A_+$, i.e. we introduce our vector-like gauge field A, we thus see that

$$\partial j^{Axial} \equiv \partial_- j_+ - \partial_+ j_- = k \star F^{Vector} \ . \tag{26}$$

Hence, the divergence of the axial current contains the field-strength of the vector gauge field.

If $k_L \neq k_R$ then the vector gauge field itself is already anomalous in the ultraviolet but the analog of the equation (27) depends on the arbitrary constant k which we have no preferred way of fixing. One gets a somewhat nice answer if one takes $k = \frac{1}{2}(k_L + k_R)$,

$$\partial j^{Vector} \equiv \partial_- j_+ + \partial_+ j_- = (k_L - k_R) \star F^{Vector} \ . \tag{27}$$

but for other choices one does not even get the field strength on the right hand side.

Coming back to (25), we can denote the infrared limit of the function a

$$\lim_{p^2\to 0} a(p^2/M^2) = \gamma \, .$$

Then we readily identify

$$k_L^{IR} = k_L^{UV} + \gamma \, , \qquad k_R^{IR} = k_R^{UV} + \gamma \, .$$

The difference between the current algebra levels remains invariant under the flow. This is a derivation of the anomaly matching first suggested by 't Hooft.

2 Gravitational Anomaly Matching in $d = 2$

We reconsider the two-point functions of the energy momentum tensor and allow for non-parity invariant pieces (which we previously omitted). This leads to a derivation of the gravitational anomaly matching in $d = 2$, which is, morally speaking, analogous to the other anomaly matching conditions of 't Hooft.

In the study of the two-point functions of the energy-momentum tensor in general 2d theories, we always insist on invariance under the Poincaré group, and, related to that, the symmetry of the energy-momentum tensor in its two indices. We will see that in general diffeomorphism invariance cannot be maintained, but we will insist on maintaining Poincaré invariance in flat space.

We start from 6 functions

$$\langle T_{++}T_{++}\rangle = p_+^4 f_1(p^2) \, , \qquad \langle T_{++}T_{+-}\rangle = -p_+^2 p^2 f_2(p^2) \, , \qquad \langle T_{++}T_{--}\rangle = p^4 f_3(p^2) \, ,$$

$$\langle T_{+-}T_{+-}\rangle = p^4 f_4(p^2) \, , \qquad \langle T_{--}T_{+-}\rangle = -p_-^2 p^2 f_5(p^2) \, , \qquad \langle T_{--}T_{--}\rangle = p_-^4 f_6(p^2) \, .$$

Conservation reads $p_+T_{--} + p_-T_{+-} = 0$ and $p_-T_{++} + p_+T_{+-} = 0$. We also used the notation $p^2 = p_+p_-$.

One finds after imposing conservation the following

$$f_1 = f_2 = f_3 = f_4 = f_5 = f_6 \, .$$

Now one has to be careful in the interoperation of these equations. They are only true at separated points. More precisely, the actual equations that one finds are $p^2 f_1 = p^2 f_2$ etc. These Ward identities should be interpreted to hold by the imaginary part of f since the imaginary part always obeys all the classical Ward identities. Therefore, we can always add $\delta(p^2)$ to the imaginary part and still satisfy these equations. This corresponds to adding pieces that look like $1/p^2$ to the functions f_i.

Therefore the **most general** admissible solution takes the following form

$$\langle T_{++}T_{++}\rangle = p_+^4\left(\frac{1}{p^2}f(p^2/M^2)+\frac{\lambda_1}{p^2}\right) , \qquad \langle T_{++}T_{+-}\rangle = -p_+^2p^2\left(\frac{1}{p^2}f(p^2/M^2)+\frac{\lambda_2}{p^2}\right) ,$$

$$\langle T_{++}T_{--}\rangle = p^4\left(\frac{1}{p^2}f(p^2/M^2)+\frac{\lambda_3}{p^2}\right) , \qquad \langle T_{+-}T_{+-}\rangle = p^4\left(\frac{1}{p^2}f(p^2/M^2)+\frac{\lambda_4}{p^2}\right) ,$$

$$\langle T_{--}T_{+-}\rangle = -p_-^2p^2\left(\frac{1}{p^2}f(p^2/M^2)+\frac{\lambda_5}{p^2}\right) , \qquad \langle T_{--}T_{--}\rangle = p_-^4\left(\frac{1}{p^2}f(p^2/M^2)+\frac{\lambda_6}{p^2}\right) .$$

The coefficients $\lambda_{2,3,4,5}$ respect the conservation equations at separated points We can view these coefficients as arbitrary scheme choice that we can make. More below.

We can choose without loss of generality the function f such that

$$\lim_{p\to\infty} f(p^2/M^2) = 0 . \tag{28}$$

This means that f scales with a positive power of the mass, i.e. f reacts to the relevant perturbation in the UV. Then, we can identify

$$\lambda_1 = c_L^{UV} , \qquad \lambda_6 = c_R^{UV} .$$

So we write

$$\langle T_{++}T_{++}\rangle = \frac{1}{p^2}p_+^4\left(f(p^2/M^2)+c_L^{UV}\right) , \qquad \langle T_{++}T_{+-}\rangle = -p_+^2\left(f(p^2/M^2)+\lambda_2\right) ,$$

$$\langle T_{++}T_{--}\rangle = p^2\left(f(p^2/M^2)+\lambda_3\right) , \qquad \langle T_{+-}T_{+-}\rangle = p^2\left(f(p^2/M^2)+\lambda_4\right) ,$$

$$\langle T_{--}T_{+-}\rangle = -p_-^2\left(f(p^2/M^2)+\lambda_5\right) , \qquad \langle T_{--}T_{--}\rangle = \frac{1}{p^2}p_-^4\left(f(p^2/M^2)+c_R^{UV}\right) .$$

We see that if $c_L^{UV}\neq c_R^{UV}$ we **cannot** obey the conservation Ward identities for the real parts. Some local terms violate the Ward identities. But if $c_L^{UV}=c_R^{UV}\equiv c$ then we can choose $\lambda_2=\lambda_3=\lambda_4=\lambda_5=c$ and then the Ward identities for conservation are obeyed both by the real and the imaginary part. Then one has a further theorem that this can be done in all correlation functions.

If $c_L^{UV}\neq c_R^{UV}$, is there a preferred way to choose $\lambda_{2,3,4,5}$? well, actually, we can just choose $\lambda_{2,3,4,5}=0$. This has the added value that it preserves $T_{+-}=0$ at coincident points! Under normal circumstances we prefer to sacrifice the trace in order to save diffeomorphisms. But if the central charges are different, diffeomorphisms cannot be saved anyway so we can at least save the trace. Therefore we arrive at our final result

$$\langle T_{++}T_{++}\rangle = \tfrac{1}{p^2}p_+^4\left(f(p^2/M^2)+c_L^{UV}\right) , \qquad \langle T_{++}T_{+-}\rangle = -p_+^2 f(p^2/M^2 ,$$

$$\langle T_{++}T_{--}\rangle = p^2 f(p^2/M^2) , \qquad \langle T_{+-}T_{+-}\rangle = p^2 f(p^2/M^2) ,$$

$$\langle T_{--}T_{+-}\rangle = -p_-^2 f(p^2/M^2) , \qquad \langle T_{--}T_{--}\rangle = \frac{1}{p^2}p_-^4\left(f(p^2/M^2)+c_R^{UV}\right) .$$

We can now easily prove the generalized 't Hooft matching condition. When we go to the infrared we have the general expansion

$$p^2/M^2 << 1: \quad f(p^2/M^2) = \delta + \mathscr{O}(p^2/M^2)$$

This allows us to define the infrared central charges

$$c_L^{IR} = c_L^{UV} + \delta \ , \qquad c_R^{IR} = c_R^{UV} + \delta$$

and we see that the difference remains invariant.

In some sense, this means that massive degrees of freedom are "parity even," meaning that they do not disturb the imbalance between right moving and left moving massless degrees of freedom.

3 Monotonicity Theorems in $d = 2$

The formalism developed above allows to prove some monotonicity theorems. We first review the approach of Zamolodchikov and then generalize it slightly and discuss the monotonicity theorems for $U(1)$ currents [23]. We also discuss the c-theorem in theories with a gravitational anomaly (a discussion of this problem appeared in [51]).

We consider non-scale invariant theories, i.e. theories where there is some conformal field theory at short distances, CFT_{uv}, and some other conformal field theory (that could be trivial) at long distances, CFT_{ir}. Let us study the correlation functions of the stress tensor in such a case, following [21]. To avoid having to discuss contact terms (which were very important above) we switch to position space. We begin by rewriting (10) in position space.

In terms of the complex coordinate $z = x^1 + ix^2$ the conservation equations are $\partial_{\bar{z}} T_{zz} = -\partial_z T$, $\partial_z T_{\bar{z}\bar{z}} = -\partial_{\bar{z}} T$, where T stands for the trace of the energy-momentum tensor. We can parameterize the most general two point functions consistent with the isometries of $\mathbb{R}^2$

$$\begin{aligned}
\langle T_{zz}(z) T_{zz}(0)\rangle &= \frac{F(z\bar{z},M)}{z^4} \ , \\
\langle T(z) T_{zz}(0)\rangle &= \frac{G(z\bar{z},M)}{z^3\bar{z}} \ , \\
\langle T(z) T(0)\rangle &= \frac{H(z\bar{z},M)}{z^2\bar{z}^2} \ .
\end{aligned} \tag{29}$$

In the above M stands for some generic mass scale of the theory. As we have seen in our analysis above (10), we know that the conservation equation should bring down the number of independent functions to two. Indeed, we find the following relations $\dot{F} = -\dot{G} + 3G$, $\dot{H} - 2H = -\dot{G} + G$, where $\dot{X} \equiv |z^2| \frac{dX}{d|z|^2}$, leaving two real undermined functions (remember that G and F are complex).

Using these relations one finds that the combination $C \equiv F - 2G - 3H$ satisfies the following differential equation

$$\dot{C} = -6H \ . \tag{30}$$

Exercise 7: Show (30)

However, since H is positive definite, the equation above means that C decreases monotonically as we increase the distance. Let us now identify C at very short and very long distances. At very short and very long distances it is described by the appropriate quantities in the corresponding conformal field theories. As we have explained above, in conformal field theory G, H are contact terms, hence can be neglected as long we do not let the operators collide. On the other hand, $F \sim c$. (It is easy to verify that F is sensitive only to the combination $b + d$ as defined in (85). Hence, it is only sensitive to c.)

This shows that C is a monotonic decreasing function that starts from c_{UV} and flows to c_{IR}. Since the anomalies c_{UV}, c_{IR} are defined inherently in the corresponding conformal field theories, this means that the space of 2d CFTs admits a natural foliation and the renormalization group flow can proceed in only one direction in this foliation. No cycles of the renormalization group are allowed.

One can think of c as a measure of degrees of freedom of the theory. In simple renormalization group flows it is easy to understand that c should decrease since we merely integrate out some massive particles. However, there are many highly non-trivial renormalization group flows where there are emergent degrees of freedom, and the result that

$$c_{UV} > c_{IR} \tag{31}$$

is a strong constraint on the allowed emergent degrees of freedom.

We can integrate the equation (30) to obtain a certain sum rule

$$c_{UV} - c_{IR} \sim \int d\log|z^2| H \sim \int d^2z |z^2| \langle T(z)T(0)\rangle > 0 \ . \tag{32}$$

Since c can also be understood as the path integral over the two-sphere, the inequality (31) can also be interpreted as a statement about the partition function of the massive theory on $\mathbb{S}^2$.

It is useful to consider non-parity invariant theories, where one can also prove a c-theorem. This is done using the formalism we developed in the previous sections. It follows directly that if we consider the function $\frac{\partial^2}{\partial p^2}\langle T_{+-}(p)T_{+-}(-p)\rangle$, our analysis shows that in the infrared this approaches the constant δ while in the ultraviolet it goes to zero. Therefore we can just write an explicit expression for the zero momentum Fourier mode

$$\delta = -\int d^2x x^2 \langle T_{+-}(x)T_{+-}(0)\rangle \ .$$

This immediately shows that in unitary theory $\delta < 0$ and hence we have $c_L^{IR} < c_L^{UV}$, $c_R^{IR} < c_R^{UV}$. Thus the c-theorem holds even if there is a gravitational anomaly.

A very similar argument allows to prove that the total current algebra level decreases along renormalization group flows (while the difference between left and right remains constant, as we explained above)

Consider the integral

$$\int d^2xx^2\langle\partial_- j_+(x)\partial_- j_+(0)\rangle \ .$$

The operator $\partial_- j_+$ is redundant in the deep UV and so the integral above converges. But in the conformal field theory itself we have $\langle\partial_- j_+(x)\partial_- j_+(0)\rangle = k_L\Box\delta^{(2)}(x)$, which gives a contribution to the integral above. We can remove it by cutting a little region away from $x=0$. Therefore we get

$$k_L^{IR} = -\int_{x\in\mathbb{R}^2_\varepsilon} d^2xx^2\langle\partial_- j_+(x)\partial_- j_+(0)\rangle + k_L^{UV} \ , \tag{33}$$

with $\mathbb{R}^2_\varepsilon$ being the plane without some small disc around the origin. Since the integral is manifestly positive we have $\gamma<0$, which shows that the current algebra level can just decrease (both the left and right handed current algebra levels decrease by the same amount).

Note that during the flow, the axial symmetry is violated not just by an anomaly, but operatorially. For compact groups, the integral in (33) would have to yield an integer. Therefore, theories with $k=1$ can only flow to theories without global symmetries in the infrared (all the charged states are massive).

4 Trace Anomalies in Four Dimensions

We saw that in two dimensions the natural monotonic property of the RG evolution was tightly related to the trace anomaly in two dimensions. In three dimensions the main role was played by the three-sphere partition function (there are no trace anomalies in three dimensions).

In four dimensions there are two trace anomalies and the monotonic property of renormalization group flows concerns again with these anomalies. The anomalous correlation function is now

$$\langle T_{\mu\nu}(q)T_{\rho\sigma}(p)T_{\gamma\delta}(-q-p)\rangle$$

And again, like in our analysis in two dimensions, there are contact-terms which are necessarily inconsistent with $T_\mu^\mu=0$. In four dimensions it turns out that there are two independent trace anomalies. Introducing a background metric field we have

$$T_\mu^\mu = aE_4 - cW^2 \ , \tag{34}$$

where $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the Euler density and $W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the Weyl tensor squared. These are called the a- and c-anomalies, respectively.

It was conjectured in [42] (and shortly after studied extensively in perturbation theory in [43],[44]) that if the conformal field theory in the ultraviolet, CFT$_{UV}$, is deformed and flows to some CFT$_{IR}$ then

$$a_{UV} > a_{IR} \ . \tag{35}$$

The four-dimensional c-anomaly does not satisfy such an inequality (this can be seen by investigating simple examples) and also the free energy density divided by the appropriate power of the temperature does not satisfy such an inequality.

In two and three dimensions we have seen that the quantities satisfying inequalities like (31), (91) are computable from the partition functions on $\mathbb{S}^2$ and $\mathbb{S}^3$, respectively. Similarly, in four dimensions, since the four-sphere is conformally flat, the partition function on $\mathbb{S}^4$ selects only the a-anomaly. Indeed,

$$\partial_{\log r} \log Z_{\mathbb{S}^4} = -\int_{\mathbb{S}^4} \sqrt{g}\langle T^\mu_\mu \rangle = -a \int_{\mathbb{S}^4} \sqrt{g} E_4 = -64\pi^2 a \ .$$

In the formula above r stands for the radius of $\mathbb{S}^4$. Real scalars contribute to the anomalies $(a,c) = \frac{1}{90(8\pi)^2}(1,3)$, Weyl fermion: $(a,c) = \frac{1}{90(8\pi)^2}(11/2,9)$, and a $U(1)$ gauge field: $(a,c) = \frac{1}{90(8\pi)^2}(62,36)$.

In four dimensions, a free gauge field is a conformal field theory because (83) is traceless. So the problems with the free gauge field that we have discussed at great length in three dimensions do not exist in four dimensions. (Similar issues arise for the free two-form gauge theory, but this does not appear naturally in the ultraviolet of interesting models in four dimensions.)

Chapter 3
Conformal Anomalies and the Zamolodchikov Metric

Now that we have some experience with anomalies, we can discuss a more refined type of trace anomaly. This anomaly is very important for the physics of conformal manifolds. Let us for a moment consider the definition (4) more carefully, and in arbitrary dimension. The Zamolodchikov metric on the conformal manifold is given by

$$\langle O_I(x) O_J(0) \rangle_\lambda = \frac{g_{IJ}(\lambda)}{x^{2d}} \ , \tag{36}$$

where $0 \neq x \in R^d$. In momentum space the two-point function (36) takes the form

$$\langle O_I(p) O_J(-p) \rangle_\lambda \sim g_{IJ}(\lambda) \begin{cases} p^d & d = 2n+1 \\ p^{2n} \log\left(\frac{\mu^2}{p^2}\right) & d = 2n \end{cases} \tag{37}$$

with $n \in \mathbb{N}$. Thus, if we rescale μ the even-dimensional result will change by a polynomial in p^2 (delta function in position space). It follows that the separated points correlation function is covariant under such rescaling while the coincident points correlation function is not covariant. The appearance of such a logarithm in conformal field theories signifies a conformal anomaly, which manifests itself as a non-vanishing contribution to the trace of the stress-energy tensor. By promoting the exactly marginal couplings λ^I to spacetime dependent background fields $\lambda^I(x)$, such that they act as sources of the exactly marginal operators $O_I(x)$, one can detect a contribution to the trace anomaly of the schematic form

$$T^\mu_\mu \supset g_{IJ} \lambda^I \Box^{\frac{d}{2}} \lambda^J \ . \tag{38}$$

The precise action of the derivatives in the formula above will be determined below.

The trace anomaly $\left\langle T^\mu_\mu \right\rangle$ can be derived from the variation of the free energy, $\delta_\sigma \log Z$, under an infinitesimal Weyl rescaling,

$$\delta_\sigma \gamma_{\mu\nu} = 2\delta\sigma\, \gamma_{\mu\nu} \ , \tag{39}$$

where $\gamma_{\mu\nu}$ is the spacetime metric. $\delta_\sigma \log Z$ must be local in $\gamma_{\mu\nu}$, $\delta\sigma$ and λ, and its form is constrained by the Wess-Zumino consistency condition, which is simply the statement that Weyl transformations commute; $\delta_\sigma\delta_{\sigma'}\log Z = \delta_{\sigma'}\delta_\sigma \log Z$. It also needs to be invariant under coordinate transformations in spacetime, and under coordinate transformations in the conformal manifold. If we have some symmetry that is respected by the class of regulators we consider (supersymmetry for example), we will require $\delta_\sigma \log Z$ to preserve this symmetry as well.

In addition, $\delta_\sigma \log Z$ is defined only up to terms that can be written as $\delta_\sigma W$ for some local functional W (which also needs to respect the symmetry constraints described above), as such terms can be removed by adding local counterterms to the free energy (in other words these terms can be removed by choosing an appropriate regulator and therefore they do not contribute to the anomaly, which, by definition, cannot be removed with any choice of regulator). Thus, in order to find the allowed form of the anomaly one needs to solve a cohomology problem.

In two-dimensional CFTs, the local functional that produces the Weyl variation of (37) is

$$\delta_\sigma \log Z \sim \int d^2x \sqrt{\gamma}\,\delta\sigma g_{IJ}\partial_\mu\lambda^I\partial^\mu\lambda^J \tag{40}$$

This manifestly satisfies the Wess-Zumino consistency condition. One can think of this anomaly as a non-linear sigma model in the space of theories.

In four-dimensional CFTs, the local functional that produces the Weyl variation of (37) is much more complicated because the Laplacian squared is not Weyl-covariant [5]

$$\delta_\sigma \log Z \supset \frac{1}{192\pi^2}\int d^4x\sqrt{\gamma}\,\delta\sigma\left(g_{IJ}\hat{\Box}\lambda^I\hat{\Box}\lambda^J - 2\,g_{IJ}\partial_\mu\lambda^I\left(R^{\mu\nu}-\frac{1}{3}\gamma^{\mu\nu}R\right)\partial_\nu\lambda^J\right), \tag{42}$$

where coordinate invariance in $\mathscr{M}$ requires introducing a connection

$$\hat{\Box}\lambda^I = \Box\lambda^I + \Gamma^I_{JK}\partial^\mu\lambda^J\partial_\mu\lambda^K\,, \tag{43}$$

and the Wess-Zumino consistency condition forces this connection to be the Christoffel connection on $\mathscr{M}$:

$$\Gamma^I_{JK} = g^{IR}\left(\partial_K g_{RJ} + \partial_J g_{RK} - \partial_R g_{JK}\right)\,. \tag{44}$$

Exercise 8: Show that coordinate invariance in the space of coupling constants together with the Wess-Zumino consistency conditions force the anomaly to take the above form.

The anomaly (42) needs to be added to the well-known conformal anomalies:[6]

---

[5] The normalization conventions here will be such that the exactly marginal deformation is

$$S \to S + \frac{1}{\pi^2}\lambda^I\int d^4x\, O_I(x)\,. \tag{41}$$

The convention we use for $R_{\mu\nu\rho\sigma}$ is $[\nabla_\mu,\nabla_\nu]V_\rho = R_{\mu\nu\rho\sigma}V^\sigma$.

[6] E_4 denotes the Euler density and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

$$\delta_\sigma \log Z \supset \frac{1}{16\pi^2}\int d^4x\,\sqrt{\gamma}\,\delta\sigma\left(cC^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - aE_4\right)\,, \tag{45}$$

which do not depend on the coordinates λ^I.

Let us now discuss the $\mathcal{N}=2$ superconformal manifold. We will assume that the superconformal theory is regulated in a way that preserves diffeomorphism invariance and $\mathcal{N}=2$ supersymmetry, i.e. we assume that the physics at coincident points is supersymmetric and diffeomorphism invariant.[7] The assumption above constrains the way the anomaly and the allowed counterterms can depend on the parameters of the theory and on the spacetime geometry. A convenient way to implement these constraints is to derive the anomaly and the counterterms as supergravity invariants that are constructed from supergravity multiplets. For this sake the parameters of the theory and of the geometry need to be embedded into supergravity multiplets.

According to equation (7) the exactly marginal operators are integrals over half superspace of chiral and antichiral superfields with $\Delta = r = 2$. Thus, the corresponding couplings need to be realized as bottom components of chiral and antichiral superfields, Λ^I and $\bar{\Lambda}^{\bar{I}}$, with $\Delta = r = 0$.[8] In addition, the Weyl variation $\delta\sigma$ is embedded in the bottom component of the chiral Weyl superfield $\delta\Sigma$ (see, e.g. [13] for details) and the integration measure $\sqrt{\gamma}$ is promoted to the density measure superfield E. In terms of these superfields, the supersymmetrization of the anomaly (42) is given by the superspace integral

$$\delta_\Sigma \log Z \supset \frac{1}{192\pi^2}\int d^4x\,d^4\theta\,d^4\bar{\theta}\,E(\delta\Sigma + \delta\bar{\Sigma})K(\Lambda^I,\bar{\Lambda}^{\bar{I}})\,. \tag{46}$$

When this integral is expanded in components, one finds (among many other terms) the anomaly (42) with

$$g_{I\bar{J}} = \partial_I\partial_{\bar{J}}K\,. \tag{47}$$

We therefore conclude that for $\mathcal{N}=2$ SCFTs, the Zamolodchikov metric is Kähler. This statement, which is true also for $\mathcal{N}=1$ SCFTs, was proven in [1] using superconformal Ward identities. The present derivation will allow to go much further.

Expanding (46) in components while keeping only the bottom component of Λ^I, $\bar{\Lambda}^{\bar{I}}$ and the metric background (setting the auxiliary fields in the gravity multiplet to zero) one ends up with the following anomaly:

[7] Note that we cannot assume that the coincident points physics is conformal invariant since this would contradict (37).

[8] Note that $\Lambda^I(x,\theta) = \lambda^I$ (with constant λ^I) is consistent with the supersymmetry variations of a chiral multiplet, and that substituting this in $\int d^4x\,d^4\theta\,\Lambda^I(x,\theta)\Phi_I(x,\theta) + \text{c.c.}$ one gets equation (7) back. After constructing the anomaly and counterterms in terms of the superfields $\Lambda^I(x,\theta)$, $\bar{\Lambda}^{\bar{I}}(x,\bar{\theta})$ we substitute the constant background values. We do a similar thing with the geometry background parameters.

$$
\begin{aligned}
\delta_\Sigma \log Z \supset \frac{1}{96\pi^2}\int d^4x\sqrt{\gamma}\Bigg\{ & \delta\sigma \mathscr{R}_{I\bar{K}J\bar{L}}\nabla^\mu\lambda^I\nabla_\mu\lambda^J\nabla^\nu\bar{\lambda}^{\bar{K}}\nabla_\nu\bar{\lambda}^{\bar{L}} \\
& + \delta\sigma g_{I\bar{J}}\left(\hat{\Box}\lambda^I\hat{\Box}\bar{\lambda}^{\bar{J}} - 2\left(R^{\mu\nu}-\frac{1}{3}R\gamma^{\mu\nu}\right)\nabla_\mu\lambda^I\nabla_\nu\bar{\lambda}^{\bar{J}}\right) \\
& + \frac{1}{2}K\Box^2\delta\sigma + \frac{1}{6}K\nabla^\mu R\nabla_\mu\delta\sigma + K\left(R^{\mu\nu}-\frac{1}{3}\gamma^{\mu\nu}R\right)\nabla_\mu\nabla_\nu\delta\sigma \\
& - 2g_{I\bar{J}}\nabla^\mu\lambda^I\nabla^\nu\bar{\lambda}^{\bar{J}}\nabla_\mu\nabla_\nu\delta\sigma + ig_{I\bar{J}}\left(\hat{\nabla}^\mu\hat{\nabla}^\nu\lambda^I\nabla_\nu\bar{\lambda}^{\bar{J}} - \hat{\nabla}^\mu\hat{\nabla}^\nu\bar{\lambda}^{\bar{J}}\nabla_\nu\lambda^I\right)\nabla_\mu\delta a \\
& - \frac{i}{2}\left(\hat{\nabla}_I\hat{\nabla}_J K\nabla^\mu\lambda^I\nabla_\mu\lambda^J - \hat{\nabla}_{\bar{I}}\hat{\nabla}_{\bar{J}}K\nabla^\mu\bar{\lambda}^{\bar{I}}\nabla_\mu\bar{\lambda}^{\bar{J}} + \nabla_I K\hat{\Box}\lambda^I - \nabla_{\bar{I}}K\hat{\Box}\bar{\lambda}^{\bar{I}}\right)\Box\delta a \\
& + i\left(R^{\mu\nu}-\frac{1}{3}R\gamma^{\mu\nu}\right)\left(\nabla_I K\nabla_\mu\lambda^I - \nabla_{\bar{I}}K\nabla_\mu\bar{\lambda}^{\bar{I}}\right)\nabla_\nu\delta a\Bigg\}\,.
\end{aligned}
\tag{48}
$$

where, as before, the hats denote covariant derivatives with respect to coordinate transformations in the conformal manifold, $\mathscr{R}_{I\bar{J}K\bar{L}} = \partial_I\partial_{\bar{J}}g_{K\bar{L}} - g^{M\bar{N}}\partial_I g_{K\bar{L}}\partial_{\bar{J}}g_{M\bar{N}}$, and $\delta\sigma + i\delta a$ is the bottom component of $\delta\Sigma$. Note that the anomaly (42) appears in the second line.

Setting λ, $\bar{\lambda}$ to constants, we remain with a non-vanishing contribution:

$$
\begin{aligned}
\delta_\Sigma \log Z \supset & \frac{1}{96\pi^2}K(\lambda,\bar{\lambda})\int d^4x\sqrt{\gamma}\left(\frac{1}{2}\Box^2\delta\sigma + \frac{1}{6}\nabla^\mu R\nabla_\mu\delta\sigma + \left(R^{\mu\nu}-\frac{1}{3}\gamma^{\mu\nu}R\right)\nabla_\mu\nabla_\nu\delta\sigma\right) \\
= & \delta_\sigma\left(\frac{1}{96\pi^2}K(\lambda,\bar{\lambda})\int d^4x\sqrt{\gamma}\left[\frac{1}{8}E_4 - \frac{1}{12}\Box R + f(\lambda,\bar{\lambda})C^2\right]\right)\,,
\end{aligned}
\tag{49}
$$

where $f(\lambda,\bar{\lambda})$ is an arbitrary function on $\mathscr{M}$, E_4 is the Euler density and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

Note that this expression is not cohomologically trivial. The second line in (49) is written as a Weyl variation of a local term, but this is not a supersymmetric local term. Thus, this contribution cannot be removed with a supersymmetric regulator. In the next section we will show that, as a result of this term, the sphere partition function has a universal (i.e. regularization independent) content - it computes the Kähler potential on the superconformal manifold.

Chapter 4
Sphere Partition Functions

5 General Observations

Any conformal field theory on R^d can be placed on S^d using the stereographic projection. Since this map is a conformal transformation we can obtain correlation functions in R^d from the corresponding correlation functions in S^d by applying the inverse map. The sphere is compact and therefore the theory on the sphere is free from infrared divergences. Since the sphere is locally equivalent to R^d, the ultraviolet divergences on the sphere are the same as in flat space. To understand whether the sphere partition function is physical, we need to examine these ultraviolet divergences carefully.

The ultraviolet divergences are classified by diffeomorphism invariant local terms of dimension $\leq d$ constructed from the background fields λ^i (i.e. the space-time dependent coupling constants) and the space-time metric g_{mn}. One starts from the following general ansatz for the partition function as a function of the point λ^i on the conformal manifold:

$$d = 2n: \qquad Z_{S^{2n}} = A_1(\lambda^i)(r\Lambda_{UV})^{2n} + A_2(\lambda^i)(r\Lambda_{UV})^{2n-2} + \ldots + A_n(\lambda^i)(r\Lambda_{UV})^2 + A(\lambda^i)\log(r\Lambda_{UV}) + F_{2n}(\lambda^i)\ . \tag{50}$$

$$d = 2n+1: \qquad Z_{S^{2n+1}} = B_1(\lambda^i)(r\Lambda_{UV})^{2n+1} + B_2(\lambda^i)(r\Lambda_{UV})^{2n-1} + \ldots + B_{n+1}(\lambda^i)(r\Lambda_{UV}) + F_{2n+1}(\lambda^i)\,, \tag{51}$$

where r is the radius of S^d. The power-law divergent terms correspond to counterterms of the type

$$\Lambda_{UV}^{2n-2k+2}\int d^{2n}x\sqrt{g}A_k(\lambda^i)R^{k-1}$$

in even dimensions and

$$\Lambda_{UV}^{2n-2k+3}\int d^{2n+1}x\sqrt{g}B_k(\lambda^i)R^{k-1}$$

in odd dimensions. Therefore, all the power-law divergent terms in (50) and (51) can be tuned to zero in the continuum limit.

In even dimensions, the sphere partition function has a logarithmic dependence on the radius (see (50)), which cannot be canceled by a local counterterm. It is associated to the Weyl anomaly. The variation of the partition function under a Weyl transformation with parameter σ contains $\int d^{2n}x\sqrt{g}\,\sigma A(\lambda^i)E_{2n}$, where E_{2n} is the Euler density (the other terms in the Weyl anomaly vanish on the sphere, because they are Weyl-invariant see [28]). However, this violates the Wess-Zumino consistency condition unless $A(\lambda^i) = A$, i.e. $A(\lambda^i)$ is a constant. This is then identified with the usual A-type anomaly which we explained previously. The coefficient of the logarithm is therefore independent of exactly marginal deformations.

Finally, the function $F_{2n}(\lambda^i)$ in (50) is ambiguous, as it can be removed by the local counterterm

$$\int d^{2n}x\sqrt{g}F_{2n}(\lambda^i)E_{2n} \tag{52}$$

In summary, we have shown that the only physical data in the continuum limit of $Z_{S^{2n}}$ is the A-anomaly, which is independent of the exactly marginal parameters.

For odd dimensions, absent additional restrictions on the counterterms, we have seen above that all the B_i are ambiguous and can be tuned to zero (a logarithmic term is absent because one cannot write an appropriate local anomaly polynomial in odd dimensions.) Importantly, however, there is no counterterm for $F_{2n+1}(\lambda^i)$ in (51). More precisely, the only conceivable dimensionless counterterm would be a gravitational Chern-Simons term

$$\int C(\lambda)\,\Omega^{(2n+1)} \tag{53}$$

but because of coordinate invariance it cannot depend on the λ^i, i.e. $C(\lambda) =$ constant. Moreover, it has to have an imaginary coefficient due to CPT symmetry. Hence, the real part of $F_{2n+1}(\lambda^i)$ is an unambiguous physical observable and is calculable in any choice of regularization scheme that preserves coordinate invariance.[9]

A comment about F_{2n+1}: It measures the finite entanglement entropy across a S^{2n-1} in $\mathbb{R}^{2n,1}$ [29]. The entanglement entropy provides another way to see that the finite part in even dimensions is ambiguous while in odd dimensions it is physical. Indeed, it is straightforward to write finite local counterterms on the entangling surface of even-dimensional spheres, while in odd dimensions this is impossible. For example, in $d = 3$, the entangling surface is a circle, and the finite counterterm $\int_{S^1} |\kappa| dl$ is forbidden because the absolute value renders it nonlocal, while without the absolute value symbol it is not consistent with the vacuum being a pure state.

We can show that $F_{2n+1}(\lambda_i)$ is constant on the conformal manifold $\mathscr{S}$. Start at an arbitrary point on the conformal manifold and expand to second order

[9] The imaginary part is more subtle. Only its fractional part is well defined. See, for example, the discussions in [24][25]. We will not comment any further on the imaginary part of F_{2n+1}.

$$Z[\lambda] \simeq Z(0)\left[1+\frac{\lambda^i}{\pi^{d/2}}\int d^dx\sqrt{g}\langle O_i\rangle + \frac{\lambda^i\lambda^j}{2\pi^d}\int d^dx\sqrt{g}\int d^dy\sqrt{g}\big\langle O_i(x)O_j(y)\big\rangle\right]$$

Conformal Ward identities on the sphere imply that $\langle O_i\rangle = 0$. Since this is true at every point on the conformal manifold, we conclude that the continuum conformal field theory sphere partition function is independent of exactly marginal parameters. This argument cannot be repeated in even dimensions because there are various finite counterterms. Indeed, we will see below that the sphere partition function may depend on exactly marginal parameters.

Now comes the most crucial observation for the application that we present below. We have argued that the finite part of the S^{2n} partition function is scheme dependent because one may add the counter-term (52). However, in some special circumstances, e.g. if the theory has a bigger symmetry group, we may not want to allow arbitrary renormalization group transformations but only those that preserve some symmetry.

It turns out that in $d = 4$ $\mathscr{N} = 2$ theories, which we discuss below, this counter-term (52) is forbidden[10] and thus the finite part of the partition function becomes truly physical and meaningful.

6 An Application for $d = 4$ $\mathscr{N} = 2$ Theories

In the recent years the exact computation of some supersymmetric observables in $\mathscr{N} = 2$ theories on S^4 became possible due to the technique of supersymmetric localization in which the path integral is reduced to a finite dimensional integral. In particular, sphere partition functions for Lagrangian $\mathscr{N} = 2$ theories (not necessarily conformal) can be computed exactly, including all perturbative and instanton contributions [15]. In this section we will show that the sphere partition function for $\mathscr{N} = 2$ SCFTs computes the Kähler potential (47) on the superconformal manifold. This was proved in [7, 8, 10]. Here, we will follow [10], in which this statement was derived from the anomaly (49).

According to equation (49), the sphere partition function, when regulated in a supersymmetry preserving fashion, contains the contribution:[11]

$$\log Z_{S^4} \supset \frac{1}{96\pi^2}K(\lambda,\bar{\lambda})\int_{S^4} d^4x\sqrt{\gamma}\left(\frac{1}{8}E_4 - \frac{1}{12}\Box R\right) = \frac{1}{12}K(\lambda,\bar{\lambda})\,. \tag{54}$$

An additional contribution comes from the usual a-anomaly. Together, the two contributions give

$$Z_{S^4} = \left(\frac{r}{r_0}\right)^{-4a} e^{K(\lambda,\bar{\lambda})/12}\,, \tag{55}$$

[10] Unless it is purely holomorphic

[11] We dropped the Weyl tensor since it vanishes on the conformally flat sphere.

where r is the radius of the sphere and r_0 a scheme dependent scale. Thus, the sphere partition function computes the Kähler potential on the superconformal manifold. This is reminiscent of a known result in two-dimensional theories. For $d=2$, $\mathcal{N}=(2,2)$ SCFTs the Zamolodchikov metric is Kähler and the sphere partition function, which has been computed using localization in [3, 6], computes the Kähler potential on the superconformal manifold [9, 12].

Note that the Kähler potential is defined up to a holomorphic ambiguity,

$$K(\lambda,\bar{\lambda}) \to K(\lambda,\bar{\lambda}) + F(\lambda) + \bar{F}(\bar{\lambda})\,. \tag{56}$$

This ambiguity in $\log Z_{S^4}$ is due to the existence of a supersymmetric counterterm that depends on an arbitrary holomorphic function of λ^I. This counterterm can be constructed from the supergravity invariant

$$\int d^4x d^4\theta\, \mathscr{E} F(\Lambda)\left(\Xi - W^{\alpha\beta}W_{\alpha\beta}\right) + \text{c.c.} \tag{57}$$

Here $\mathscr{E}$ is a chiral density superfield. The chiral superfields Ξ and $W_{\alpha\beta}$ can be found in [13]. In the sphere geometry background, and with the substitution $\Lambda^I(x,\theta)=\lambda^I$, this evaluates to $F(\lambda)+\bar{F}(\bar{\lambda})$ (up to a numerical coefficient). This counterterm was first constructed from $\mathcal{N}=2$ supergravity in [8].

As mentioned above, (54) cannot be removed by an $\mathcal{N}=2$ supersymmetric counterterm and therefore the sphere partition function has a universal meaning in $\mathcal{N}=2$ SCFTs. If we only assume that the regularization scheme preserves $\mathcal{N}=1$ supersymmetry we would have a counterterm that depends on a general function of λ and $\bar{\lambda}$.[12] Thus, $\mathcal{N}=1$ supersymmetry of the regulator is not enough to give a universal meaning to Z_{S^4}. For the same reason the λ-dependence of the sphere partition function of $\mathcal{N}=1$ SCFTs or of non-supersymmetric CFTs is regularization scheme dependent. The only universal contribution to the sphere partition function of a non-supersymmetric CFT is the contribution due to the conformal anomaly a, which is independent of the exactly marginal couplings.

As an example for the computation of the Zamolodchikov metric using equation (55), consider an $SU(2)$ gauge theory with 4 hypermultiplets in the fundamental representation. This theory is superconformal, with one exactly marginal parameter $\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}$, where g is the Yang-Mills coupling and θ is the theta angle. The sphere partition function can be computed using localization, and one finds:

$$Z_{S^4}(\tau,\bar{\tau}) = \int_{-\infty}^{\infty} da\, e^{-4\pi \operatorname{Im}\tau\, a^2}(2a)^2 \frac{H(2ia)H(-2ia)}{[H(ia)H(-ia)]^4}|Z_{\text{inst}}(a,\tau)|^2\,, \tag{58}$$

where $H(x)$ is given in terms of the Barnes G-function by $H(x)=G(1+x)G(1-x)$, and Z_{inst} is Nekrasov's instanton partition function on the Omega background [14]. By expanding the integrand in powers of g^2 we can compute Z_{S^4} to any order in perturbation theory. We can also include instanton corrections, up to any instanton number. It is then straightforward to compute the Zamolodchikov metric via

[12] See section 4 of [7].

$g_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} \log Z_{S^4}$.[13] The perturbative result for the metric is:

$$g_{\tau\bar{\tau}} = \frac{3}{8}\frac{1}{(\mathrm{Im}\tau)^2} - \frac{135\zeta(3)}{32\pi^2}\frac{1}{(\mathrm{Im}\tau)^4} + \frac{1575\zeta(5)}{64\pi^3}\frac{1}{(\mathrm{Im}\tau)^5} + \mathscr{O}\left(\frac{1}{(\mathrm{Im}\tau)^6}\right) . \quad (59)$$

The first two terms in this result were checked against an explicit, two-loop, Feynman diagrams computation in [2]. The one-instanton correction for the perturbative result is given by

$$g^{\text{1-inst}}_{\tau\bar{\tau}} = \cos\theta\, e^{-\frac{8\pi^2}{g^2}} \left(\frac{3}{8}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{3}{16\pi}\frac{1}{(\mathrm{Im}\tau)^3} - \frac{135\zeta(3)}{32\pi^2}\frac{1}{(\mathrm{Im}\tau)^4} + \mathscr{O}\left(\frac{1}{(\mathrm{Im}\tau)^5}\right)\right) . \quad (60)$$

[13] Here we dropped the factor of 12.

Chapter 5
Another Look at Monotonicity Theorems

Imagine any renormalizable QFT (in any number of dimensions) and set all the mass parameters to zero. The extended symmetry includes the full conformal group. If the number of space-time dimensions is even then the conformal group has trace anomalies. If the number of space-time dimensions is of the form $4k+2$, there may also be gravitational anomalies. We will keep ignoring gravitational anomalies here.

Upon introducing the mass terms, one violates conformal symmetry *explicitly*. Thus, in general, the conformal symmetry is violated both by trace anomalies and by an operatorial violation of the equation $T^{\mu}_{\mu}=0$ in flat space-time. The latter violation can always be removed by letting the coupling constants transform. Indeed, replace every mass scale M (either in the Lagrangian or associated to some cutoff) by $Me^{-\tau(x)}$, where $\tau(x)$ is some background field (i.e. a function of space-time). Then the conformal symmetry of the Lagrangian is restored if we accompany the ordinary conformal transformation of the fields by a transformation of τ. To linear order, $\tau(x)$ always appears in the Lagrangian as $\sim\int d^dx\ \tau T^{\mu}_{\mu}$. Setting $\tau=0$ one is back to the original theory, but we can also let τ be some general function of space-time. The variation of the path integral under such a conformal transformation that also acts on $\tau(x)$ is thus fixed by the anomaly of the conformal theory in the ultraviolet. This idea allows us to study some questions about general RG flows using the constraints of conformal symmetry. We will sometimes refer to τ as the dilaton.

Consider integrating out all the high energy modes and flow to the deep infrared. Since we do not integrate out the massless particles, the dependence on τ is regular and local. As we have explained, the dependence on τ is tightly constrained by the conformal symmetry. Since in even dimensions the conformal group has trace anomalies, these must be reproduced by the low energy theory. The conformal field theory at long distances, CFT_{IR}, contributes to the trace anomalies, but to match to the defining UV theory, the τ functional has to compensate precisely for the difference between the anomalies of the conformal field theory at short distances, CFT_{UV}, and the conformal field theory at long distances, CFT_{IR}.

Let us see how these ideas are borne out in two-dimensional renormalization group flows. Let us study the constraints imposed by conformal symmetry on action functionals of τ (which is a background field). An easy way to analyze these constraints is to introduce a fiducial metric $g_{\mu\nu}$ into the system. Weyl transformations act on the dilaton and metric according to $\tau \to \tau+\sigma$, $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$. If the Lagrangian for the dilaton and metric is Weyl invariant, upon setting the metric to be flat, one finds a conformal invariant theory for the dilaton. Hence, the task is to classify local diff$\times$ Weyl invariant Lagrangians for the dilaton and metric background fields.

It is convenient to define $\hat{g}_{\mu\nu} = e^{-2\tau}g_{\mu\nu}$, which is Weyl invariant. At the level of two derivatives, there is only one diff$\times$Weyl invariant term: $\int \sqrt{\hat{g}}\hat{R}$. However, this is a topological term, and so it is insensitive to local changes of $\tau(x)$. Therefore, if one starts from a diff$\times$Weyl invariant theory, upon setting $g_{\mu\nu} = \eta_{\mu\nu}$, the term $\int d^2x(\partial\tau)^2$ is absent because there is no appropriate local term that could generate it.

The key is to recall that unitary two-dimensional theories have a trace anomaly

$$T^{\mu}_{\mu} = -\frac{c}{24\pi}R\ . \tag{61}$$

One must therefore allow the Lagrangian to break Weyl invariance, such that the Weyl variation of the action is consistent with (61). The action functional which reproduces the two-dimensional trace anomaly is

$$S_{WZ}[\tau, g_{\mu\nu}] = \frac{c}{24\pi}\int \sqrt{g}\left(\tau R + (\partial\tau)^2\right)\ . \tag{62}$$

Exercise 9: Check that this satisfies the Wess-Zumino condition.

We see that even though the anomaly itself disappears in flat space (61), there is a two-derivative term for τ that survives even after the metric is taken to be flat. This is of course the familiar Wess-Zumino term for the two-dimensional conformal group. (It also appears as the Liouville or linear dilaton action in the context of conformal field theory.)

Consider now some general two-dimensional RG flow from a CFT in the UV (with central charge c_{UV} and a CFT in the IR (with central change c_{IR}). Replace every mass scale according to $M \to Me^{-\tau(x)}$. We also couple the theory to some background metric. Performing a simultaneous Weyl transformation of the dynamical fields and the background field $\tau(x)$, the theory is non-invariant only because of the anomaly $\delta_\sigma S = \frac{c_{UV}}{24}\int d^2x\sqrt{g}\sigma R$. Since this is a property of the full quantum theory, it must be reproduced at all scales. An immediate consequence of this idea is that also in the deep infrared the effective action should reproduce the transformation $\delta_\sigma S = \frac{c_{UV}}{24}\int d^2x\sqrt{g}\sigma R$. At long distances, one obtains a contribution c_{IR} to the anomaly from CFT$_{IR}$, hence, the rest of the anomaly must come from an explicit Wess-Zumino functional (62) with coefficient $c_{UV} - c_{IR}$. In particular, setting the background metric to be flat, we conclude that the low energy theory must contain a term

$$\frac{c_{UV} - c_{IR}}{24\pi}\int d^2x(\partial\tau)^2\ . \tag{63}$$

Note that the coefficient of this term is universally proportional to the difference between the anomalies and it does not depend on the details of the flow. Higher-derivative terms for the dilaton can be generated from local diff×Weyl invariant terms, and there is no a priori reason for them to be universal (that is, they may depend on the details of the flow, and not just on the conformal field theories at short and long distances).

Zamolodchikov's theorem that we reviewed in the first chapter follows directly from (63). Indeed, from reflection positivity we must have that the coefficient of the term (63) is positive, and thus, the inequality is established.

We can be more explicit. The coupling of τ to matter must take the form $\tau T^{\mu}_{\mu} + \cdots$, where the corrections have more τs. To extract the two-point function of τ with two derivatives we must use the insertion τT^{μ}_{μ} twice. (Terms containing τ^2 can be lowered once, but they do not contribute to the two-derivative term in the effective action of τ.) As a consequence, we find that

$$\begin{aligned}\left\langle e^{\int \tau T^{\mu}_{\mu} d^2x}\right\rangle &= \cdots + \frac{1}{2}\int\int \tau(x)\tau(y)\langle T^{\mu}_{\mu}(x)T^{\mu}_{\mu}(y)\rangle d^2x d^2y + \cdots \\ &= \cdots + \frac{1}{4}\int \tau(x)\partial_\rho\partial_\sigma\tau(x)\left(\int (y-x)^\rho(y-x)^\sigma\langle T^{\mu}_{\mu}(x)T^{\mu}_{\mu}(y)\rangle d^2y\right)d^2x + \cdots \;.\end{aligned} \tag{64}$$

In the final line of the equation above, we have concentrated entirely on the two-derivative term. It follows from translation invariance that the y integral is x-independent

$$\int (y-x)^\rho(y-x)^\sigma\langle T^{\mu}_{\mu}(x)T^{\mu}_{\mu}(y)\rangle d^2y = \frac{1}{2}\eta^{\rho\sigma}\int y^2\langle T^{\mu}_{\mu}(0)T^{\mu}_{\mu}(y)\rangle d^2y \;. \tag{65}$$

To summarize, one finds the following contribution to the dilaton effective action at two derivatives

$$\frac{1}{8}\int d^2x\tau\Box\tau\int d^2y y^2\langle T(y)T(0)\rangle \;. \tag{66}$$

According to (63), the expected coefficient of $\tau\Box\tau$ is $(c_{UV} - c_{IR})/24\pi$, and so by comparing we obtain

$$\Delta c = 3\pi\int d^2y y^2\langle T(y)T(0)\rangle \;. \tag{67}$$

As we have already mentioned, $\Delta c > 0$ follows from reflection positivity (which is a property of unitary theories). Equation (67) precisely agrees with the classic results about two-dimensional flows (32).

7 A New Variation on the k-Theorem

Consider A CFT as above and couple the currents to background gauge fields as

$$\int d^2x\,(j_+A_- + j_-A_+) \;.$$

Then the generating functional is naively invariant under both transformations

$$A \to A + dv \; ,$$

$$A \to A + *d\omega \; .$$

The theory is naively invariant under both of these transformations because the current and its Poincaré dual are both conserved. To avoid complications we take $k_L = k_R$. Then we can preserve the vector current and only violate the axial current by an anomaly (27). So we assume that we regularize the theory in a V-preserving fashion and thus the partition functional satisfies

$$\delta_v \log Z = 0 \; , \qquad \delta_\omega \log Z = k \int d^2x \; \omega \varepsilon^{\mu\nu} F^{Vector}_{\mu\nu} \; . \tag{68}$$

Actually, the Wess-Zumino consistency conditions are not trivial because $\delta_\omega \varepsilon^{\mu\nu} F_{\mu\nu} = \Box\omega$. The commutator of two axial transformations therefore gives

$$[\delta_{\omega_1}, \delta_{\omega_2}] \log Z = \int d^2x \omega_1 \Box \omega_2 - 1 \leftrightarrow 2 \; .$$

This vanishes by integration by parts.

Imagine now an RG flows that preserves j^V. In the deep infrared, and in the deep UV, j^A becomes conserved as well. But it is not conserved throughout the flow.

To re-instate the axial symmetry throughout the flow, we introduce an axion field π that transforms under Axial transformations but is invariant under Vector transformations

$$A: \; \pi \to \pi + \omega \; , \qquad V: \; \pi \to \pi \; .$$

As in the proof of the c-theorem above, we now need to look for a *local* functional that satisfies

$$\delta_\omega S_{local}[A, \pi] = \int d^2x \; \omega \varepsilon^{\mu\nu} F^{Vector}_{\mu\nu} \; , \qquad \delta_v S_{local}[A, \pi] = 0 \; .$$

Such a local functional is given by

$$S_{local} = \int d^2x \left(\pi \star F^V + \pi \Box \pi \right) \; .$$

In particular, we see that we need to add a Wess-Zumino-Witten term for the axion.

(For supersymmetric theories this would not be surprising since both π and the dilaton sit in the conformal compensator of supergravity so a kinetic term for the dilaton set by the c-anomaly is a SUSY friend of a kinetic term for the axion set by the k-anomaly.)

In any case, since π has to have a positive definite kinetic term and since the coefficient of S_{local} is multiplied by $k_{UV} - k_{IR}$ for anomaly matching to work, we obtain the k-theorem. In more detail, the coupling in the action of π looks like $\int d^2x \pi \partial \cdot j^A + \ldots$ and therefore to second order in *pi* the generating functional is

$$\int d^2k\pi(k)\pi(-k)\langle\partial\cdot j^A(k)\partial\cdot j^A(-k)\rangle$$

and we are interested in the expnansion of $\langle\partial\cdot j^A(k)\partial\cdot j^A(-k)\rangle$ to second order in k, i.e. $\frac{\partial^2}{\partial k^2}\langle\partial\cdot j^A(k)\partial\cdot j^A(-k)\rangle\Big|_{k=0}$. This is given just by

$$k_{UV}-k_{IR}=\int d^2xx^2\langle\partial\cdot j^A(x)\partial\cdot j^A(0)\rangle>0\ .$$

So in the end one obtains a result that coincides with the more direct derivation through two-point functions.

The very significant advantage of this method is that it is also applicable to 2d defects, while the more straightforward method is not (at least not with the current understanding of defects)!

8 The Monotonicity Theorem in Four Dimensions

One starts by classifying local diff$\times$Weyl invariant functionals of τ and a background metric $g_{\mu\nu}$. Again, we demand invariance under $g_{\mu\nu}\longrightarrow e^{2\sigma}g_{\mu\nu}$, $\tau\longrightarrow\tau+\sigma$. We will often denote $\hat{g}=e^{-2\tau}g_{\mu\nu}$. The combination $\hat{g}$ transforms as a metric under diffeomorphisms and is Weyl invariant.

The most general theory up to (and including) two derivatives is:

$$f^2\int d^4x\sqrt{-\det\hat{g}}\left(\Lambda+\frac{1}{6}\hat{R}\right)\ , \tag{69}$$

where we have defined $\hat{R}=\hat{g}^{\mu\nu}R_{\mu\nu}[\hat{g}]$. Since we are ultimately interested in the flat-space theory, let us evaluate the kinetic term with $g_{\mu\nu}=\eta_{\mu\nu}$. Using integration by parts we get

$$S=f^2\int d^4xe^{-2\tau}(\partial\tau)^2\ . \tag{70}$$

One can use the field redefinition $\Psi=1-e^{-\tau}$ to rewrite this as

$$S=f^2\int d^4x\Psi\Box\Psi\ . \tag{71}$$

One can also study terms in the effective action with more derivatives. With four derivatives, one has three independent (dimensionless) coefficients

$$\int d^4x\sqrt{-\hat{g}}\left(\kappa_1\hat{R}^2+\kappa_2\hat{R}^2_{\mu\nu}+\kappa_3\hat{R}^2_{\mu\nu\rho\sigma}\right)\ . \tag{72}$$

It is implicit that indices are raised and lowered with $\hat{g}$. Recall the expressions for the Euler density $\sqrt{-g}E_4$ and the Weyl tensor squared $E_4=R^2_{\mu\nu\rho\sigma}-4R^2_{\mu\nu}+R^2$,$W^2_{\mu\nu\rho\sigma}=R^2_{\mu\nu\rho\sigma}-2R^2_{\mu\nu}+\frac{1}{3}R^2$ We can thus choose instead of the basis of local terms (72) a different parameterization

$$\int d^4x\sqrt{-\hat{g}}\left(\kappa_1'\hat{R}^2+\kappa_2'\hat{E}_4+\kappa_3'\hat{W}^2_{\mu\nu\rho\sigma}\right) . \tag{73}$$

We immediately see that the κ_2' term is a total derivative. If we set $g_{\mu\nu}=\eta_{\mu\nu}$, then $\hat{g}_{\mu\nu}=e^{-2\tau}\eta_{\mu\nu}$ is conformal to the flat metric and hence also the κ_3' term does not play any role as far as the dilaton interactions in flat space are concerned. Consequently, terms in the flat space limit arise solely from $\hat{R}^2$. A straightforward calculation yields

$$\int d^4x\sqrt{-\hat{g}}\hat{R}^2\Big|_{g_{\mu\nu}=\eta_{\mu\nu}}=36\int d^4x\left(\Box\,\tau-(\partial\tau)^2\right)^2\sim\int d^4x\frac{1}{(1-\Psi)^2}\left(\Box\Psi\right)^2 . \tag{74}$$

So far we have only discussed diff$\times$Weyl invariant terms in four-dimensions, but from the two-dimensional re-derivation of the c-theorem we have shown above, we anticipate that the anomalous functional will play a key role.

The most general anomalous variation one needs to consider takes the form $\delta_\sigma S_{anomaly}=\int d^4x\sqrt{-g}\sigma\left(cW^2_{\mu\nu\rho\sigma}-aE_4\right)$. The question is then how to write a functional $S_{anomaly}$ that reproduces this anomaly. (Note that $S_{anomaly}$ is only defined modulo diff$\times$Weyl invariant terms.) Without the field τ one must resort to non-local expressions, but in the presence of the dilaton one has a local action.

It is a little tedious to compute this local action, but the procedure is straightforward in principle. We first replace σ on the right-hand side of the anomalous variation with τ

$$S_{anomaly}=\int d^4x\sqrt{-g}\tau\left(cW^2_{\mu\nu\rho\sigma}-aE_4\right)+\cdots . \tag{75}$$

While the variation of this includes the sought-after terms, as the $\cdots$ indicate, this cannot be the whole answer because the object in parenthesis is not Weyl invariant. Hence, we need to keep fixing this expression with more factors of τ until the procedure terminates. Note that $\sqrt{-g}W^2_{\mu\nu\rho\sigma}$, being the square of the Weyl tensor, is Weyl invariant, and hence we do not need to add any fixes proportional to the c-anomaly This makes the c-anomaly "Abelian" in some sense.

The "non-Abelian" structure coming from the Weyl variation of E_4 is the key to our construction. The a-anomaly is therefore quite distinct algebraically from the c-anomaly.

The final expression for $S_{anomaly}$ is (see [49], where the anomaly functional was presented in a form identical to what we use in this note)

$$\begin{aligned}S_{anomaly}=&-a\int d^4x\sqrt{-g}\left(\tau E_4+4\big(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\big)\partial_\mu\tau\partial_\nu\tau-4(\partial\tau)^2\Box\tau+2(\partial\tau)^4\right)\\&+c\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma} .\end{aligned} \tag{76}$$

Exercise 10 (lengthy): reproduce this result.

Note that even when the metric is flat, self-interactions of the dilaton survive. This is analogous to what happens with the Wess-Zumino term in pion physics when the background gauge fields are set to zero and this is also what we saw in two dimensions.

Setting the background metric to be flat we thus find that the non-anomalous terms in the dilaton generating functional are

$$\int d^4x \left(\alpha_1 e^{-4\tau} + \alpha_2 (\partial e^{-\tau})^2 + \alpha_3 \left(\Box\tau - (\partial\tau)^2\right)^2\right) , \tag{77}$$

where α_i are some real coefficients.

The a-anomaly has a Wess-Zumino term, leading to the additional contribution

$$S_{WZ} = 2(a_{UV} - a_{IR}) \int d^4x \left(2(\partial\tau)^2\Box\tau - (\partial\tau)^4\right) . \tag{78}$$

The coefficient is universal because the total anomaly has to match (as we have explained in detail in two dimensions).

We see that if one knew the four-derivative terms for the dilaton, one could extract $a_{UV} - a_{IR}$. A clean way of separating this anomaly term from the rest is achieved by rewriting it with the variable $\Psi = 1 - e^{-\tau}$. Then the terms in (77) become

$$\int d^4x \left(\alpha_1 \Psi^4 + \alpha_2 (\partial\Psi)^2 + \frac{\alpha_3}{(1-\Psi)^2}(\Box\Psi)^2\right) , \tag{79}$$

while the WZ term (78) is

$$S_{WZ} = 2(a_{UV} - a_{IR}) \int d^4x \left(\frac{2(\partial\Psi)^2\Box\Psi}{(1-\Psi)^3} + \frac{(\partial\Psi)^4}{(1-\Psi)^4}\right) . \tag{80}$$

We see that if we consider background fields Ψ which are null ($\Box\Psi = 0$), α_3 disappears and only the last term in (80) remains. Therefore, by computing the partition function of the QFT in the presence of four null insertions of Ψ one can extract directly $a_{UV} - a_{IR}$.

Indeed, consider all the diagrams with four insertions of a background Ψ with momenta k_i, such that $\sum_i k_i = 0$ and $k_i^2 = 0$. Expanding this amplitude, $\mathscr{A}$, to fourth order in the momenta k_i, one finds that the momentum dependence takes the form $s^2 + t^2 + u^2$ with $s = 2k_1 \cdot k_2$, $t = 2k_1 \cdot k_3$, $u = 2k_1 \cdot k_4$. Our effective action analysis shows that the coefficient of $s^2 + t^2 + u^2$ is directly proportional to $a_{UV} - a_{IR}$.

In fact, one can even specialize to the so-called forward kinematics, choosing $k_1 = -k_3$ and $k_2 = -k_4$. Then the amplitude is only a function of $s = 2k_1 \cdot k_2$. $a_{UV} - a_{IR}$ can be extracted from the s^2 term in the expansion of the amplitude around $s = 0$. Continuing s to the complex plane, there is a branch cut for positive s (corresponding to physical states in the s-channel) and negative s (corresponding to physical states in the u-channel). There is a crossing symmetry $s \leftrightarrow -s$ so these branch cuts are identical.

To calculate the imaginary part associated to the branch cut we utilize the optical theorem. The imaginary part is manifestly positive definite. Using Cauchy's theorem we can relate the low energy coefficient of s^2, $a_{UV} - a_{IR}$, to an integral over the branch cut. Fixing all the coefficients one finds

$$a_{UV} - a_{IR} = \frac{1}{4\pi} \int_{s>0} \frac{Im\mathscr{A}(s)}{s^3} \ . \tag{81}$$

As explained, the imaginary part $Im\mathscr{A}(s)$ can be evaluated by means of the optical theorem, and it is manifestly positive. Since the integral converges by power counting (and thus no subtractions are needed), we conclude

$$a_{UV} > a_{IR} \ . \tag{82}$$

Note the difference between the ways positivity is established in two and four dimensions. In two dimensions, one invokes reflection positivity of a two-point function (reflection positivity is best understood in *Euclidean* space). In four dimensions, the Wess-Zumino term involves four dilatons, so the natural positivity constraint comes from the forward kinematics (and hence, it is inherently *Minkowskian*).

Let us say a few words about the physical relevance of $a_{UV} > a_{IR}$. Such an inequality constrains severely the dynamics of quantum field theory, and the allowed renormalization group flows. In favorable cases can be used to establish that some symmetries must be broken or that some symmetries must be unbroken. In a similar fashion, if a system naively admits several possible dynamical scenarios one can use $a_{UV} > a_{IR}$ as an additional handle.

The work of [35] shows that a can be also obtained from the entanglement entropy across an S^2, i.e. $A = D^3$ in our previous notation. But so far it has not been shown that (82) can be derived by manipulating the entanglement entropy and the inequalities it obeys.

Chapter 6
Three Dimensions

Three-dimensional QFT is directly relevant for understanding interesting classical second order phase transitions (boiling water, He_3 etc), as well as quantum critical points that appear condensed matter physics. Finally, it is a useful playground for confinement and other non-perturbative aspects of quantum field theory.

Here we will discuss the symmetries of fixed points, S^3 partition functions, monotonicity of RG flows, connection to entanglement entropy, and supersymmetry.

9 Conformal Invariance

There are interesting examples of continuum theories with infinite correlation length but no conformal symmetry. Consider the free gauge field in three dimensions

$$S = \frac{1}{2e^2}\int d^3x F_{\mu\nu}^2$$

One would think that the theory is scale invariant because we can assign the gauge field dimension $1/2$ and thus e would be dimensionless.

However, the only conceivable, conserved, gauge invariant, energy-momentum tensor we could write is

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}\eta_{\mu\nu}F^2 \ . \tag{83}$$

Exercise 10: Show that this energy-momentum tensor is conserved.

It is not traceless in three-dimensions $T^{\mu}_{\mu} = \frac{1}{4}F^2$. It is possible to prove that a local scale current has to be of the form

$$\Delta_\mu = x^\nu T_{\mu\nu} - V_\mu \ , \tag{84}$$

where V_μ is called the virial current. This is conserved if an only if $T^\mu_\mu = \partial^\nu V_\nu$, i.e. the energy-momentum tensor is a gradient of a well-defined operator. In our case, the best we can do is to write a non-gauge invariant scale current

$$\Delta_\mu = x^\nu T_{\mu\nu} - \frac{1}{2} F_{\mu\nu} A^\nu \tag{85}$$

which is conserved.

It is not a good scale current because it is not gauge invariant. However, the charge

$$\Delta = \int d^2 x \Delta_0 \tag{86}$$

is gauge invariant if we assume all the fields decay sufficiently rapidly.

Exercise 11: Show that the scale current (85) **is conserved and show that the associated charge is gauge invariant.**

So we conclude that the theory has no scale current, but a scale charge exists. One can also show that the currents of special conformal transformations don't exist, but in this case, also the associated charges does not exist. So free 3d QED is an example of a theory that is unitary, scale invariant, but not conformal.

It is not really known whether this theory is an exception or there are more theories of this sort (it would be very interesting to find a unitary interacting example – it is fair to say that most people believe such theories do not exist). This free counterexample to the idea that scale invariance+unitarity imply conformal invariance has been discussed in many places, see for example [22],[26] and references therein.

10 A More Careful Look into Free 3d QED (for advanced students)

The discussion above is essentially correct, but there is a more precise way to phrase it. In three dimensions a free gauge field and a free scalar field are completely equivalent

$$\partial_\mu \phi = \varepsilon_{\mu\nu\rho} F^{\nu\rho} \ . \tag{87}$$

The Bianchi identity corresponds to the Klein-Gordon equation.

One has to distinguish the case that the gauge symmetry acting on A_μ is compact from the non-compact case. In the former, one can have a nontrivial flux of the magnetic field through two-cycles. That means that ϕ is a periodic scalar, so that it can wind through the dual one-cycle. If we decompactify the gauge symmetry, the period of the scalar goes to zero. So the scalar dual to a gauge field with noncompact gauge symmetry is of zero radius (this is a scalar without a zero mode).

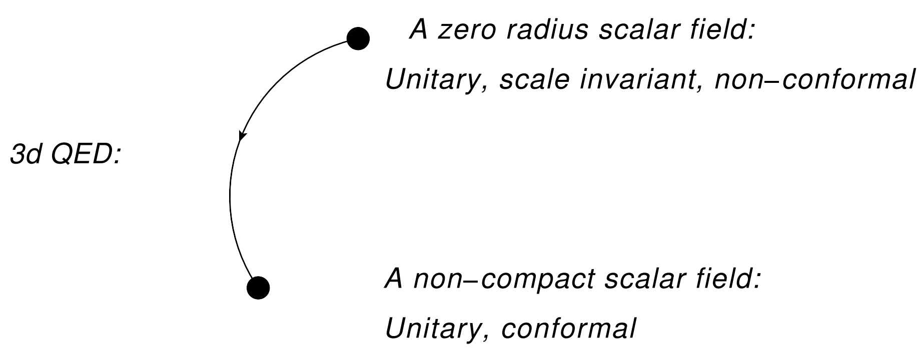
The clash between having a conserved energy momentum tensor and conformal invariance is clearly visible in the language of ϕ. A traceless energy-momentum tensor for ϕ would take the form

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 + \frac{1}{8}(\eta_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu)\phi^2 \tag{88}$$

Exercise 12: Show that this is conserved and traceless.

However, this energy-momentum tensor is inconsistent with the periodicity of ϕ. One can naturally interpret the period of ϕ as some dimensionful parameter (of dimension 1/2). In the far UV, the periodicity goes to zero (since it has mass dimension) and therefore one finds that the UV theory is a scalar without a zero mode. This is a unitary scale but non-conformally invariant theory. In the infrared the scalar effectively de-compactifies. Then, the energy momentum tensor (88) becomes admissible. Hence, the infrared theory is just an ordinary, conformal, non-compact free scalar field.

One can therefore summarize the free 3d gauge field model in the following diagram:



One can test this interpretation of the 3d QED model by, for example, computing the entanglement entropy on a disk. One indeed finds that this approaches infinity logarithmically in the UV (we will discuss why this has to be the case below) and a finite value, identical to that of a free non-compact scalar field, in the infrared [27].

11 S^3 Partition Functions

An interesting question is whether there exists a function in three dimensions satisfying something similar to (31). The problem consists of identifying a candidate quantity that could satisfy such an inequality and then proving that it indeed does so.

There are various ways to define quantities in higher-dimensional field theories that share some common features with c. For example, in conformal theories in two dimensions c is equivalent to the free energy density of the system, divided by the appropriate power of the temperature. One could define a similar object in higher-dimensional field theories. However, one quickly finds that it is not monotonic [31]. This already shows that inequalities such as (31) are quite delicate, and they fail if one chooses to measure the number of effective degrees of freedom in the wrong way (albeit a very intuitive and seemingly natural way).

Progress on the problem of identifying a candidate quantity generalizing (31) happened quite recently [32],[33]. The conjecture arose independently from studies in AdS/CFT and from studies of $\mathcal{N}=2$ supersymmetric 3d theories.

Any conformal field theory on $\mathbb{R}^3$ can be canonically mapped to a theory on the curved space $\mathbb{S}^3$. This is because $\mathbb{S}^3$ is stereographically equivalent to flat space (thus the metric on $\mathbb{S}^3$ is conformal to $\mathbb{R}^3$). In three dimensions there are no trace anomalies, and hence the partition function over $\mathbb{S}^3$ has no logarithms of the radius. (This should be contrasted with the situation in two dimensions, (23).)

Consider

$$Z_{\mathbb{S}^3} = \int [d\Phi] e^{-\int_{\mathbb{S}^3} \mathscr{L}(\Phi)} \ . \tag{89}$$

This is generally divergent and takes the form (for a three-sphere of radius a)

$$\log Z_{\mathbb{S}^3} = c_1(\Lambda a)^3 + c_2(\Lambda a) + F \ . \tag{90}$$

Terms with inverse powers of Λ are dropped since they are not part of the continuum theory. They can be tuned away by adding the cosmological constant counter-term and the Einstein-Hilbert counter-term. However, no counter term can remove the finite part F.[14]

Imagine a three-dimensional flow from some CFT_{uv} to some CFT_{ir}. Then we can (in principle) compute F_{uv} and F_{ir} via the procedure above. The conjecture is

$$F_{uv} > F_{ir} \ . \tag{91}$$

Let us outline the computation of F is simple examples. Take a free massless scalar $\mathscr{L} = \frac{1}{2}(\partial\Phi)^2$. To put it in a curved background while preserving conformal invariance (more precisely, Weyl invariance) we write in d dimensions

$$S = \frac{1}{2}\int d^3x\sqrt{g}\left((\nabla\Phi)^2 + \frac{d-2}{4(d-1)}R[g]\Phi^2\right) \ . \tag{92}$$

This coupling to the Ricci scalar is necessary to preserve Weyl invariance. Weyl invariance means that the action is invariant under rescaling the metric by any function. We achieve this by accompanying the action on the metric with some action on the fields. For the action above, Weyl invariance means that the action is invariant under

[14] More precisely, one can have the gravitational Chern-Simons term, but this cannot affect the real part of F. We disregard the imaginary part of F in our discussion.

$$g \to e^{2\sigma} g \,, \qquad \phi \to e^{-\frac{1}{2}\sigma} \phi \,. \tag{93}$$

We can now compute the partition function on the three-sphere by diagonalizing the corresponding differential operator $-\log Z_{S^3} = \frac{1}{2} \log\det\left(-\nabla^2 + \frac{1}{8}R\right)$. The Ricci scalar is related to the radius in three dimensions via $R = \frac{6}{a^2}$. The eigenfunctions are of course well known. The eigenvalues are

$$\lambda_n = \frac{1}{a^2}\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right) \,,$$

and their respective multiplicities are

$$m_n = (n+1)^2 \,.$$

The free energy on the three-sphere due to a single conformally coupled scalar is therefore

$$-\log Z_{\mathbb{S}^3} = \frac{1}{2}\sum_{n=0}^{\infty} m_n \left(-2\log(\mu_0 a) + \log\left(n + \frac{3}{2}\right) + \log\left(n - \frac{1}{2}\right)\right) \,. \tag{94}$$

We have inserted an arbitrary scale μ_0 to soak up the dependence on the radius of the sphere. Since there are no anomalies in three dimensions, we expect that there would be no dependence on μ_0 eventually.

This sum clearly diverges and needs to be regulated. We choose to regulate it using the zeta function. One finds that with this regulator $\sum_{n=0} m_n = \zeta(-2) = 0$ and therefore a logarithmic dependence on the radius is absent, as anticipated. We remain with

$$F_{scalar} = -\frac{1}{2}\frac{d}{ds}\left[2\zeta(s-2,\frac{1}{2}) + \frac{1}{2}\zeta(s,\frac{1}{2})\right] = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx 0.0638$$

One can perform a similar computation for a free massless Dirac fermion field and one finds

$$F_{fermion} = \frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2} \approx 0.219$$

The absolute value of the partition function of a massless Majorana fermion is just a half of the result above. We see that the counting of degrees of freedom is quite nontrivial.

An interesting fact is that a nonzero contribution to F arises also from *topological* degrees of freedom. This has to be contrasted with the situation in two dimensions, where c was defined through a local correlation function and hence was oblivious to topological matter. For example, let us take Chern-Simons theory associated to some gauge group G,

$$S = \frac{k}{4\pi}\int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \,, \tag{95}$$

where k is called the level. This theory has no propagating degrees of freedom. Indeed, the equation of motion is

$$0 = F = dA + A \wedge A \ ,$$

which means that the curvature of the gauge field vanishes everywhere. Such gauge fields are called flat connections. The space of flat connection on the manifold M is fixed completely by topological properties of the manifold.

The partition function of CS theory on the three sphere has been discussed in [34]. In particular, for $U(1)$ CS theory the answer is $\frac{1}{2}\log k$ while for $U(N)$ it is

$$F_{CS}(k,N) = \frac{N}{2}\log(k+N) - \sum_{j=1}^{N-1}(N-j)\log\left(2\sin\frac{\pi j}{k+N}\right) \ . \tag{96}$$

We see that the contribution from a topological sector can be in fact arbitrarily large as we take the level k to be large.

Let us now check the inequality (91) in a simple renormalization group flow. One can start from the conformal field theory described by $U(1)_k$ CS theory coupled to N_f Dirac fermions of charge 1. This is a conformal field theory because the Lagrangian has no coupling constant that can run. (The CS coefficient is discrete because it is topological in nature.) This conformal field theory is weakly coupled when $k >> 1$. Hence, the F coefficient is

$$F_{UV} \approx \frac{1}{2}\log k + N_f\left(\frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2}\right)$$

Let us now deform this by a mass term. The fermions disappear, but there is a pure CS term in the infrared with a shifted level $k \pm N_f/2$, where the sign depends on the sign of the mass term. Hence,

$$F_{IR} \approx \frac{1}{2}\log\left(k \pm N_f/2\right) \ ,$$

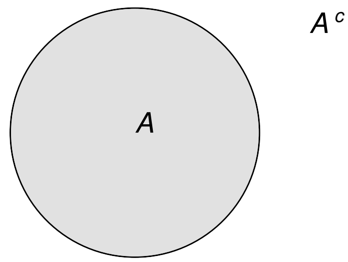
and one can convince oneself that in the regime where our analysis is valid,

$$F_{uv} > F_{ir} \tag{97}$$

holds true.

We would now like to discuss the physical interpretation of F, which is far from obvious. The partition function over S^3 does not have an obvious interpretation in terms of a Hilbert space so it is not clear what does it count.

Consider the entanglement entropy across a disk, so in the notation of the previous section, $A = D^2$, a two-dimensional disk.



In three dimensions a logarithm is not allowed in the von-Neumann entropy of a disk. (Roughly speaking, this is because there are no conformal anomalies.) In general, we expect the von-Neumann entropy of a disk of radius r to take the form

$$S_{von-Neumann} = \Lambda r + S \ , \tag{98}$$

where Λ is some UV cutoff. This linear divergence can be removed by adding a local counter-term on the boundary of the disk $\int_{S^1} d\gamma$. However, S is physical, it cannot be removed by adding any admissible local counterterm.

Exercise 13: (Challenging) suppose we add on the boundary of the disk the counter-term $\int_{S^1} d\gamma\ \kappa$ where κ is the extrinsic curvature. This would seem to make S ambiguous. Explain why the counter term $\int_{S^1} d\gamma\ \kappa$ is disallowed.

In the paper [35] it was shown that

$$F = S \ . \tag{99}$$

Therefore, one can interpret the S^3 partition function as the entanglement entropy across a disk. Additionally, one can also think of this entanglement entropy as the thermodynamic entropy in hyperbolic space [35]. We see again that monotonicity of the renormalization group flow (97) is again intimately related to the entanglement entropy.

There is not yet a conventional field theoretic proof of this inequality (91), but using the relation with the Entanglement Entropy, the inequality follows from some inequalities satisfied by the density matrix. Various issues with this construction are discussed in [37]. Of course, in all of these discussions one assumes that the entanglement entropy exists in the continuum theory.

12 Back to Free QED$_3$

Above we have computed the F associated to the theory of a free fermion and to a free conformally-coupled boson. Here we would like to go back to the subtle theory of a free gauge field in three dimensions and compute the F associated to it.

Let us start with some general comments. First, since the gauge field is dual to a scalar, one would be tempted to use (92). But remember that the gauge field is dual to a periodic scalar, so the second term in (92) is disallowed. This is of course closely related to the fact that the final term in the energy-momentum tensor (88) does not respect the gauged discrete shift symmetry. We can ask under what conditions Z_{S^3} is going to come out independent of a, the radius of S^3. Z_{S^3} would be independence of a if there exists a local, well-defined, V_μ such that

$$T_\mu^\mu = \nabla^\rho V_\rho \ . \tag{100}$$

This equation guarantees the existence of a conserved scale invariance current,

$$\Delta_\mu = x^\nu T_{\nu\mu} - V_\mu \ . \tag{101}$$

Since such a local scale current does not exist for the free gauge field, there might be an interesting dependence on a. This means that we might not be able to assign a finite F to the theory of a free gauge field.

The partition function of the free scalar field on S^3 is obtained by starting from the action

$$S = \frac{1}{2}\int d^3x\sqrt{g}\left((\nabla\phi)^2 + \alpha R\phi^2\right) \tag{102}$$

On the sphere of radius a, $R = d(d-1)/a^2$. α is a general real coefficient such that if $\alpha = 1/8$ it is conformally coupled. Only $\alpha = 0$ is consistent with the discrete shift symmetry.

The partition function as a function of α is $F = \frac{1}{2}\log\det\left(-\nabla^2 + \alpha R\right)$. To compute it one again recalls that the eigenvalues of the Laplacian on the sphere of radius a:

$$\lambda_n = \frac{1}{a^2}n(n+2)\ , \tag{103}$$

and they come with multiplicity $m_n = (n+1)^2$. Hence,

$$F = \frac{1}{2}\sum_n m_n \log\left(\lambda_n + \frac{6\alpha}{a^2}\right)\ . \tag{104}$$

Let us investigate the dependence on the a that appears inside the logarithm first. It takes the form $-\log(a)\sum_n m_n$. Employing zeta-function regularization $\sum_n n^2 = \zeta(-2) = 0$ and hence the coefficient of $\log(a)$ is zero.

We remain with

$$F = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)^2\log\left(n(n+2) + 6\alpha\right) \tag{105}$$

This is divergent as N^3 where N is some effective cutoff on the modes. This corresponds to a cosmological constant counter-term on S^3. There is also a subleading linear divergence corresponding to an Einstein-Hilbert counter-term. So the above sum needs a regulator.

We see a certain sickness for α corresponding to the periodic free scalar, $\alpha = 0$. This is easily interpreted as coming from the fact we have a non-compact scalar zero mode once we remove the curvature coupling. So this is a divergence from the infinite volume of target space. It is an infrared divergence.

We can compute the sum (105) with a zeta-function regulator. A helpful formula that we will use is

$$\sum_{n=0}^{\infty}(n+1)^2\log(n+c) = -\frac{d}{ds}\left[\zeta(s-2,c)+2(1-c)\zeta(s-1,c)+(c-1)^2\zeta(s,c)\right]\Big|_{s=0} . \tag{106}$$

Now let us consider $\alpha=0$, where the existence of a divergence due to the non-compact target space has already been noted. We take $\alpha=\frac{1}{6}\varepsilon$ to regulate things. We find

$$F[\alpha=\frac{1}{6}\varepsilon]=\frac{1}{2}\log(\varepsilon)+\frac{\log(\pi)}{2}+\frac{\zeta(3)}{4\pi^2}+\mathscr{O}(\varepsilon) . \tag{107}$$

Exercise 14: Derive (106) **and** (107)**.**

The coefficient $\frac{1}{2}$ in front of the first log would be generally replaced by the dimension of moduli space of the theory divided by two. Note that this partition function is now tending to minus infinity. When ε is small, the scalar is effectively allowed to probe distances or order $\phi\sim\varepsilon^{-1/2}R^{-1/2}$. Therefore, if we have a scalar with period *sqrt f* (where f has dimension 1), we expect that the leading logarithmic piece in the F function would be

$$F=\frac{1}{2}\log(f^{-1}R^{-1})=-\frac{1}{2}\log(fR) \tag{108}$$

We see that when the scalar has zero radius $F\to\infty$ which is completely consistent with the qualitative picture of the flow in the figure on page 12.

Since the periodic scalar and the gauge field are equivalent, we see that the free gauge field needs to be assigned infinite positive F (we again quote only the leading logarithmic term):

$$F_{S^3}^{Maxwell}=-\frac{\log(e^2a)}{2} . \tag{109}$$

This allows us an interesting reinterpretation of the result $\frac{1}{2}\log k$ of the $U(1)_k$ CS theory, quoted above (96). If we add a CS term of level k to a free Maxwell field, it picks up a mass $m\sim e^2k$ and hence the logarithm needs to be "cut-off" at spheres of radius $a^{-1}\sim e^2k$. Therefore we expect to find in the infrared $\frac{1}{2}\log k$ with a positive sign in front of the logarithm. This is precisely the result in CS theory, including the pre-factor.

We summarize that the "number of degrees of freedom" associated to a free gauge field is formally infinite. In fact, this is necessary for the F-theorem to hold: we can start from the free gauge field and flow to the topological CS term with any level k. Since the latter has $F=\frac{1}{2}\log(k)$, the only consistent F that we must assign to the free gauge field is infinite. This is somewhat unfortunate, because many interesting models whose dynamics we would like to understand (e.g. QED+flavors in 3d) start their life in the ultraviolet from free gauge fields and so the F-theorem does not easily lead to interesting bounds. However, one can still place interesting bounds by applying tricks such as in [41].

Note a conceptual difference from the *c*-theorem in two dimensions (31). In the two-dimensional case, topological degrees of freedom do not contribute. In three dimensions, one must count the topological degrees of freedom as well.

13 The S^3 Partition Function can be Computed in $\mathcal{N}=2$ Theories

If one wants to gain some information about F beyond free (or weakly coupled) models, non-perturbative methods are needed.

It turns out that if one starts from $\mathcal{N}=2$ (i.e. four supercharges) QFT in flat space ($\mathbb{R}^3$), then one can put the theory on S^3 while preserving four supercharges, which are not the same ones that are preserved in flat space. A technical requirement is that the flat space theory has at least one R-symmetry.

This statement is obvious for superconformal field theories in flat space, since they can be placed on S^3 using a stereographic transformation as above. The surprising fact is that this can be done even if the theory is non-conformal.

One can then use supersymmetric localization to compute F in many interesting examples. In particular, the inequality (91) is always satisfied. To learn about these developments see [38],[39],[40] and references therein.

Chapter 7
Scale vs Conformal Invariance

As we review in the section about two-dimensional theories, the problem in two dimensions was essentially solved long ago. In three dimensions the problem is open. In four dimensions, there has been a lot of recent work on the subject. The problem is almost solved. Let us briefly explain what we mean by "almost." The tools used to arrive at the results below are very similar to what we have discussed in the previous two sections.

Suppose one is given a unitary scale invariant theory that is not conformal. That means that there exists some V_μ such that $T^\mu_\mu = \partial^\rho V_\rho$. The theory is non-conformal if there does not exist a scalar $\mathscr{O}$ such that $V_\mu = \partial_\mu \mathscr{O}$. In fact, for the results below to hold true, one does not quite need to assume the existence of V_μ – having a well-defined scale charge Δ is sufficient. (So the results also hold for the two-form gauge theory.)

In [50] it was shown that under these assumptions, for any state vector $|X\rangle$ in the scale invariant theory, we have

$$\langle VAC|T^\mu_\mu(p)T^\mu_\mu(q)|X\rangle = 0 \,, \qquad p^2 = q^2 = 0 \,. \tag{110}$$

(The momentum of the state $|X\rangle$ is $-p-q$.) If one could prove (111) for any p,q then it would follow that $T^\mu_\mu = 0$ and thus the theory is conformal. However, one cannot hope to be able to prove that $T^\mu_\mu = 0$ because in many four-dimensional models one can improve the energy-momentum tensor.[15] The best one can hope to prove is that $T^\mu_\mu = \Box\mathscr{O}$.

In [26] the argument of [50] was generalized to prove the following

$$\forall n\,.,\forall|X\rangle.\,, \qquad \langle VAC|T^\mu_\mu(p_1)T^\mu_\mu(p_2)\ldots T^\mu_\mu(p_n)|X\rangle = 0 \,, \qquad p_i^2 = 0 \,. \tag{111}$$

It was then argued that this allows to conclude that $T = \Box\mathscr{O}$. The argument is not a proof, but it relies on some very simple and intuitive analogy with S-matrix theory. A more precise statement of what the argument shows is that, at least as far as the

[15] For example, in scalar field theory, we can add the term $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$ to the energy-momentum tensor.

theory is flat space is concerned, all its local properties must be consistent with conformal invariance (if one assumes unitarity and scale invariance).

We would like to finish with a short remark on the situation in higher dimensions: the analysis was repeated in six dimensions in [52]. Despite many encouraging facts, a proof that the Euler anomaly is monotonic does not exist yet. In five dimensions, the S^5 partition function has a physical finite part, which one should hope is monotonic in RG flows. But currently there is no argument to that effect.

Chapter 8
The Supersymmetric Index

Since the work of Witten [] it has been clear that in some situations non-perturbative computations in supersymmetric theories can be performed at weak (or even zero) coupling. Suppose we are given a supercharge Q with $\{Q,Q^\dagger\} = \Delta$ where Δ is some conserved charge. Let the Hilbert space be $\mathscr{H}$, then we may consider the following index:

$$\mathscr{I}[\mu_i] = Tr_{\mathscr{H}}\left[(-1)^F \prod_i z_i^{q_i}\right] . \tag{112}$$

From the above, it follows that only states with $\Delta = 0$ contribute to $\mathscr{I}$. The next key observation is that representations of the algebra $\{Q,Q^\dagger\} = 0$ are short compared to the case that $\Delta \neq 0$. Finally, two short representations can combine to a long representation only if they have different fermion numbers. Therefore, the trace (112) is independent of continuous coupling constants and it can be often computed at zero coupling.

However, the Witten index can be rarely computed in flat space. This is because many supersymmetric theories have a moduli space of vacua thus rendering the counting of supersymmetric vacua ill defined. Recently, it has been realized how to proceed in such cases. We can study the theory on $\mathscr{M}_{d-1}\times\mathbb{R}$ with $\mathscr{M}_{d-1}$ some compact $d-1$-dimensional manifold. Since on curved spaces one often finds that the scalar fields are coupled to curvature, one may hope that the continuous moduli space is lifted. If so, the index (112) can be computed [] and it is an interesting object to study.

Not every choice of $\mathscr{M}_{d-1}$ is consistent with preserving some supersymmetry. Let us spell out the conditions that $\mathscr{M}_{d-1}$ needs to satisfy in order for it to be consistent with unbroken supersymmetry. The proofs of these claims can be found in []. An interesting family of spaces $\mathscr{M}_3\times\mathbb{R}$ which admit unbroken SUSY generators which do not depend on time (i.e. do not depend on the coordinate of $\mathbb{R}$) is obtained by taking $\mathscr{M}_3$ to be a Seifert manifold. A Seifert manifold is simply an S^1 fibration over a Riemann surface. (Some simple examples in this class are therefore $S^2\times S^1$, S^3, and Lens spaces.) Such spaces preserve at least two supersymmetry generators δ_ζ and $\delta_{\tilde\zeta}$ of opposite R-charge. Since the supersymmetry generators are time-independent, there is no obstruction to compactifying $\mathbb{R} \to S^1$ and we can thus consider $\mathscr{M}_4 = \mathscr{M}_3 \times S^1$ with $\mathscr{M}_3$ any Seifert manifold.

The total four-dimensional space $\mathscr{M}_4 = \mathscr{M}_3 \times S^1$ is then guaranteed to be a complex manifold, and there is a holomorphic Killing vector that points in a direction which is a linear combination of the Seifert circle and the S^1 in $\mathscr{M}_3 \times S^1$. Let us call this holomorphic Killing vector K. The SUSY algebra is then

$$\{\delta_\zeta, \delta_{\tilde{\zeta}}\} = 2i\delta_K \ , \qquad \delta_\zeta^2 = \delta_{\tilde{\zeta}}^2 = 0 \ . \tag{113}$$

Let us quote another theorem that would be central for the applications to follow. Since the manifold $\mathscr{M}_3 \times S^1$ is complex, there is a moduli space of complex manifolds with this topology. This was studied by Kodaira-Spencer. One can prove that $Z_{\mathscr{M}_3 \times S^1}$, i.e. the partition function on this space, is independent of the metric that we put on $\mathscr{M}_3 \times S^1$. It only depends on the complex structure moduli.

A particularly interesting choice to make is $\mathscr{M}_3 = S^3$. Then the moduli space of complex manifolds which are topologically $\mathscr{M}_3 = S^3$ is two-complex dimensional. The partition function is independent of the metric. It only depends on these two complex numbers. If we take the field theory to be superconformal, then, since $S^{d-1} \times \mathbb{R}$ is conformally flat, the index (112) in this case can be related via radial quantization to counting local operators in $\mathbb{R}^d$ that sit in short representations of the superconformal group. We will see how the geometric interpretation of the partition function as an invariant in complex geometry and the combinatorial interpretation as an object that counts local operators are consistent.

If we denote the generator of translations along the S^1 by H, then the partition function can be interpreted combinagorially as

$$Z = Tr_{\mathscr{H}(\mathscr{M}_3)} \left[(-1)^F e^{-\beta(H - \sum_i \mu_i q_i)} \right] \ . \tag{114}$$

The length of the S^1 is $\beta \equiv 2\pi r_1 \equiv T^{-1}$. We have also allowed for various chemical potentials μ_i that couple to conserved charges q_i which commute with the SUSY generators on $\mathscr{M}_3 \times S^1$.

Let us recall Cardy's universal formula [] in two dimensions

$$\beta \to 0 : \qquad \sum_{operators} e^{-\beta\Delta} \sim e^{\frac{\pi^2 c}{3\beta}} \ . \tag{115}$$

where c is the Virasoro central charge and the spatial circle is again taken to have radius one. Equation (115) is intimately related to the modular group in two dimensions. It is therefore quite interesting that extremely similar formulae exist for the supersymmetric partition functions in $d = 4$ (and $d = 6$).

Indeed, for example in $d = 4$, we will find

$$\beta \to 0 : \qquad \sum_{operators} (-1)^F e^{-\beta(\Delta + 1/2R)} \sim e^{-\frac{16\pi^2}{3\beta}(a-c)} \ , \tag{116}$$

where we have taken the radius of the S^3 to be one, and a, c are the usual trace anomalies in four dimensions. Only operators that sit in short representations of the superconformal group contribute to the left hand side of (116). Therefore, (116) encodes a universal property of the spectral density of "heavy" BPS operators in $\mathscr{N} = 1$ SCFTs in four dimensions. Conventionally, the a- and c- anomalies are extracted from three-point functions of the energy-momentum supermultiplet. Here we see that the difference is completely fixed by the BPS spectrum.

The physics behind (116) involves rather interesting considerations in RG flows, hydrodynamics, and supersymmetry. So we will explain it now.

It is helpful to begin by recalling the construction of the usual thermal partition function of QFT (not necessarily supersymmetric). We thus consider an arbitrary QFT on the space $\mathscr{M}_3 \times S^1$ with the fermions assigned anti-periodic boundary conditions along the circle. This partition function captures the equilibrium properties of the quantum field theory at finite temperature $T = \beta^{-1} \equiv (2\pi r_1)^{-1}$. Further, let us assume the theory has a conserved $U(1)$ symmetry, with q being the corresponding charge. It is useful to introduce a background metric $g_{\mu\nu}$ that couples to the energy-momentum tensor and a background gauge field A_μ that couples to the conserved current. In order to obtain correlation functions at zero Matsubara frequency, one can reduce over the S^1 and find a *local* three-dimensional functional, $\mathscr{W}$, of the background metric and gauge field. Derivatives of $\mathscr{W}$ with respect to the background fields generate equilibrium correlation functions of the energy-momentum tensor and the conserved current.

The expansion in derivatives of $\mathscr{W}$ corresponds to the expansion in the radius of S^1 compared to the radius of $\mathscr{M}_3$. (If $\mathscr{M}_3 = \mathbb{R}^3$ then the expansion in derivatives is just the usual expansion in the 3d momentum relative to the plasma.) In detail, we take the metric and background gauge field to be

$$ds^2 = e^{2\phi}(dX^4 + a_i dx^i)^2 + h_{ij}dx^i dx^j \ , \qquad A = A_4(dX^4 + a_i dx^i) + \mathscr{A}_i dx^i \ . \quad (117)$$

The total space is topologically $\mathscr{M}_3 \times S^1$ and $i = 1,2,3$ runs over the coordinates on $\mathscr{M}_3$. All background fields are taken to be functions of only the x^i. $X^4 \simeq X^4 + \beta$ describes a circle of length β. To simplify, below we set $\phi = 0$ (it is straightforward to reintroduce ϕ).

At zeroth order in derivatives we have

$$\mathscr{W}^{(0)} = \int d^3x \sqrt{h}\, P(A_4, \beta) \quad (118)$$

with an arbitrary function P. Actually, in the absence of anomalies, A_4 would be a periodic scalar $A_4 \simeq A_4 + \frac{2\pi}{\beta}$ and so the function P should only depend on $\exp(i\beta A_4)$. P is the usual hydrodynamic pressure.

The terms which are first order in derivatives have been classified in []. They all have to be Chern-Simons-like terms.

$$\mathscr{W}^{(1)} = \frac{1}{r_1}\frac{ik_1}{4\pi}\int_{\mathscr{M}_3} \mathscr{A} \wedge da + \frac{ik_2 r_1}{4\pi}\int_{\mathscr{M}_3} A_4 \mathscr{A} \wedge d\mathscr{A} + \frac{ik_3 r_1}{4\pi}\int_{\mathscr{M}_3} A_4^2 \mathscr{A} \wedge da \ . \quad (119)$$

(In our convention for the metric, the KK photon a is dimensionless, which explains the various factors of r_1 appearing above.) Note that the coefficients k_2, k_3 are associated to field-dependent Chern-Simons terms. These are not standard terms in three-dimensional QFT because they violate gauge invariance. Here we have infinitely many KK fields in three dimensions, so such non-gauge invariant terms may arise in principle due to the need to regulate the sum over the infinitely many three-dimensional quantum fields.

Insert here digression about Chern-Simons Contact Terms in Three Dimensions, Quantization, etc.

The sum over the infinitely many KK fields has a preferred regularization. One requires that the partition function $\mathscr{W}$ satisfies the four-dimensional anomaly equation

$$A_\mu \to A_\mu + \partial_\mu \Lambda : \qquad \delta_\Lambda \mathscr{W} = -i\frac{C}{24\pi^2}\int_{\mathscr{M}_3\times S^1} \Lambda F\wedge F\ , \tag{120}$$

where C is the usual $U(1)^3$ anomaly coefficient, such that $C=1$ for a left-handed fermion of unit charge. Dimensionally reducing the right hand side of (120) over the circle, we can match with the gauge variation of (119) and find

$$k_2 = 2k_3 = -\frac{2}{3}C\ . \tag{121}$$

Note that such considerations do not fix k_1 because it multiplies a term that is invariant under small gauge transformations. So we have to focus on the term

$$\frac{1}{r_1}\frac{ik}{4\pi}\int_{\mathscr{M}_3} A\wedge da\ . \tag{122}$$

In examples we find

$$k = -\frac{1}{12}Tr(q)\ . \tag{123}$$

This relation was conjectured to hold in general in []. We will now give a very simple non-perturbative derivation of (123) in a large class of theories. A complete non-perturbative proof remains elusive, see however [].

Consider the four-dimensional theory of a massless Weyl fermion ψ_α charged under a $U(1)$ gauge field with charge e. We take the space to be topologically $\mathscr{M}_3\times S^1$, with the curvature of $\mathscr{M}_3$ much smaller than the inverse radius of the S^1 (i.e. the KK scale). The fermion is assigned anti-periodic condition along the S^1.

The dimensionally-reduced theory on $\mathscr{M}_3$ is gapped, and the spectrum of the low-energy theory on $\mathscr{M}_3$ is a tower of fermions, with masses $r_1 m_n = n - e r_1 A_4$, where $n\in\mathbb{Z}+1/2$. The tower is coupled to the three-dimensional gauge field $\mathscr{A}_i$ and also to the graviphoton a_i. Under the latter the nth particle carries charge $n\in\mathbb{Z}+1/2$. Recall the following fact about the 3d theory of a single massive fermion ψ_α with charges e_x under the $U(1)$ gauge fields A^x: upon integrating this fermion out, one generates the Chern-Simons terms

$$\mathscr{W}_{eff} = -\frac{i}{8\pi}\mathrm{sgn}(m)\int_{\mathscr{M}_3}\sum e_x e_y A^x \wedge dA^y \ . \tag{124}$$

Integrating out the nth KK fermion, we thus find (according to (124)) the following Chern-Simons terms:

$$-\frac{i}{8\pi}\mathrm{sgn}(n - er_1A_4)\int_{\mathscr{M}_3}\left(e^2\mathscr{A}\wedge d\mathscr{A} + 2e\frac{n}{r_1}\mathscr{A}\wedge da + ...\right) \ . \tag{125}$$

It is crucial that this is a correctly quantized, gauge invariant Chern-Simons term.

But now we need to sum over $n \in \mathbb{Z}+1/2$ .This sum is divergent. We will regulate it using the zeta function. We need the following three sums:

$$S_1(s,A_4) = \sum_{n\in\mathbb{Z}+1/2}\mathrm{sgn}(n - er_1A_4)|n - er_1A_4|^{-s} \ ,$$

$$S_2(s,A_4) = \sum_{n\in\mathbb{Z}+1/2}\mathrm{sgn}(n - er_1A_4)n|n - er_1A_4|^{-s} \ ,$$

evaluated at $s = 0$. For large enough s all the sums above converge. We take $er_1A_4 \in (-\frac{1}{2}, \frac{1}{2})$ for simplicity. After some algebra we find

$$S_1(s = 0, A_4) = 2er_1A_4 \ ,$$

$$S_2(s = 0, A_4) = e^2r_1^2A_4^2 + 1/12 \ ,$$

We thus find the following effective action:

$$\mathscr{W}_{fermion} = -\frac{i}{4\pi}\int_{\mathscr{M}_3}\left(e^3r_1A_4\mathscr{A}\wedge d\mathscr{A} + \left(e^3r_1A_4^2 + \frac{e}{12r_1}\right)\mathscr{A}\wedge da + ...\right) \ . \tag{126}$$

This agrees with the general ansatz for the effective action and we see that (123) holds true.

Additional comments:

- While the contribution from integrating out each individual field in the KK tower leads to a properly quantized Chern-Simons term, we see that the sum over the KK tower leads to incorrectly quantized (i.e. non-gauge invariant) Chern-Simons terms.
- Let us imagine an arbitrary Lagrangian field theory. (By that we mean that there exists a point in the space of continuous couplings such that the theory becomes free.) If the coefficient of $\int_{\mathscr{M}_3}\frac{1}{r_1}\mathscr{A}\wedge da$ had depended on any continuous couplings, we would have arrived at a contradiction because upon promoting these couplings to background fields we would have violated gauge invariance under small gauge transformations. Since there is no local four-dimensional anomaly to soak up this non-gauge invariance, any dependence on continuous coupling constants is therefore disallowed. We can thus compute the coefficient of $\int_{\mathscr{M}_3}\frac{1}{r_1}\mathscr{A}\wedge da$ at the free field theory point. Thus, the formula (123) follows

for any value of the coupling constants. (One can view this argument as a non-perturbative generalization of Coleman-Hill.)

Let us now add supersymmetry to this story. The gauge field $\mathscr{A}$ can be taken to be the R-symmetry gauge field, $\mathscr{A}^{(R)}$. Denote $v^i = -i\varepsilon^{ijk}\partial_j a_k$. In order to supersymmetrize (122) we need to find a $d=3$ $\mathscr{N}=2$ supergravity term that includes $\mathscr{A}_i^{(R)} v^i$. This is provided simply by the $\mathscr{N}=2$ Einstein-Hilbert term!

$$\mathscr{L}_{EH} = M\left(\frac{1}{2}\mathscr{R}^{(3)} - H^2 + 2v^i v_i - 2\mathscr{A}_i^{(R)} v^i\right) , \tag{127}$$

where H is some auxiliary field in the supergravity multiplet. For spaces of the form $\mathscr{M}_3 \times S^1$ it can be explicitly found by solving some Killing spinor equations.

Since the coefficient of (122) is fixed, we find that the scale M is fixed as well. Therefore, at very small β the leading contribution to the $\mathscr{M}_3 \times S^1$ partition function is

$$\beta \to 0 : \quad \log Z_{\mathscr{M}_3 \times S^1} = \frac{\pi^2 \kappa L_{\mathscr{M}_3}}{\beta} + \mathscr{O}(1) , \tag{128}$$

where

$$L_{\mathscr{M}_3} \equiv \frac{1}{24\pi^2}\int_{\mathscr{M}_3} dx^3 \sqrt{h}\left(\frac{1}{2}\mathscr{R}^{(3)} - H^2 + 2v^i v_i - 2\mathscr{A}_i^{(R)} v^i\right) , \tag{129}$$

$$\kappa = -Tr(R) . \tag{130}$$

14 Applications and Examples

An interesting example to consider is the partition function over $\mathscr{M}_4 = S_b^3 \times S^1$, where S_b^3 stands for the squashed three-sphere with parameter b. The metric is a product metric with S^1 having length β and the metric on S_b^3 being

$$ds^2_{S_b^3} = r_3^2\left[b^{-2}\cos^2\psi d\phi^2 + b^2\sin^2\psi d\chi^2 + f(\psi)^2 d\psi^2\right] ,$$

with $f(\psi) = \sqrt{b^2\cos^2\psi + b^{-2}\sin^2\psi}$. The range of the angles is $\phi, \chi \in [0, 2\pi]$, $\psi \in [0, \frac{\pi}{2}]$. For $b=1$ S_b^3 becomes the usual round sphere. The total space $S_b^3 \times S^1$ thus has the line element

$$ds^2 = r_1^2 d\theta^2 + ds^2_{S_b^3} ,$$

with $\theta \simeq \theta + 2\pi$. The four-dimensional metric above can be viewed as a Hermitian metric corresponding to a point on the moduli space of complex structures of $S^3 \times S^1$. (This moduli space is two-complex dimensional.) In order to write supersymmetric theories on this space one needs to activate the background field $H = -\frac{i}{r_3 f(\psi)}$ in addition to the metric. The background field v^i vanishes because the four-dimensional metric is a direct product. We are thus ready to compute $L_{S_b^3}$

$$L_{S^3_b} = \frac{r_3}{3}\frac{b+b^{-1}}{2} \ . \tag{131}$$

A comment on Geometry: There is the following family of metrics on $S^3 \times S^1$, all of which correspond to the same p,q:

$$ds^2 = r_1^2 d\theta^2 + r_3^2\left[b^{-2}\cos^2\psi d\phi^2 + b^2\sin^2\psi d\chi^2 + f(\psi)^2 d\psi^2\right] \ ,$$

but rather than taking $f(\psi)$ to be $f(\psi) = \sqrt{b^2\cos^2\psi + b^{-2}\sin^2\psi}$ as above, we could take *any* $f(\psi)$ which approaches b^{-1} at $\psi = \pi/2$ and b at $\psi = 0$. The background field H is given by $H = -\frac{i}{r_3 f(\psi)}$. The background field v^i vanishes. Our claims can therefore be consistent only if the integrated local term (127) does not depend of $f(\psi)$ (except for the values of $f(\psi)$ at the boundaries). Indeed, evaluating the local term we find that

$$L_{S^3_b} \sim \int_0^{\frac{\pi}{2}} d\psi\left(\frac{2}{f(\psi)}\sin(2\psi) + \frac{\partial_\psi f(\psi)}{f(\psi)^2}\cos(2\psi)\right)$$

$$= \int_0^{\frac{\pi}{2}} d\psi\,\partial_\psi\left(-\frac{1}{f(\psi)}\cos(2\psi)\right) = \frac{1}{f(\pi/2)} + \frac{1}{f(0)} = b + b^{-1} \ .$$

This means that we have found local densities that are invariant under some *subset* of the metric deformations of transverse holomorphic foliations. These local densities are therefore somewhat analogous to the familiar topological invariants in even dimensions.

For the superconfromal R-symmetry, we have

$$Tr(U(1)_R) = 16(a-c) \ . \tag{132}$$

We can thus rewrite the asymptotic form of the partition function as

$$\beta \to 0: \quad \log Z_{S^3_b \times S^1} = -\frac{8\pi^2 r_3(b+b^{-1})}{3\beta}(a-c) + \cdots \ . \tag{133}$$

Radial quantization allows us to reinterpret this partition sum as counting local operators in $\mathbb{R}^4$ that sit in short representations of the superconformal group, as explained above. The $\beta \to 0$ limit thus corresponds to counting all the BPS operators with signs.

One learns

- 1.) If $a - c < 0$ then fermionic and bosonic operators do not cancel against each other asymptotically, and the total Witten index in the space of local operators therefore diverges. The (absolute value of the) spectral density is asymptotically growing exponentially.
- 2.) If $a = c$ then there is a delicate albeit imperfect cancelation between bosonic and fermionic short representations. The spectral density does not grow exponentially asymptotically.

- 3.) If $a-c>0$ then there is a perfect cancelation between fermionic and bosonic short representations. The spectral density is asymptotically oscillatory and the bigger $a-c$ is, the more frequent the oscillations are.

Thus, the scenario that $a-c>0$ might seem unlikely or non-generic from this point of view. This could explain why it is much more difficult to construct examples with $a-c>0$, although clearly not impossible. (e.g. the free vector field of the previous subsection).

Note that when $a=c$ we find a vanishing coefficient for the three-dimensional Einstein-Hilbert term in the effective action on S^3. Examples of SCFTs with $a=c$ include the theories with $\mathcal{N}=4$ supersymmetry. This perhaps suggests that it could be impossible to complete the Einstein-Hilbert term to an action preserving the extended (off-shell) supersymmetry.

The connection between the sign of $a-c$ and the asymptotic structure of short representations is reminiscent of the Kutasov-Seiberg theorem.

15 More on the Central Charge

The central charge c appears in several other very important places. We would like to explain very briefly why c appears in the entanglement entropy of the vacuum [20]. Suppose we take $\mathbb{R}=A\cup A^c$ and $A=[a,b]$ some interval. The reduced density matrix is defined by

$$\rho_A = Tr_{A^c}(\rho_{vacuum}) \ ,$$

with ρ_{vacuum} the density matrix of the normal pure vacuum. Then, the Renyi entropies are defined by

$$S_n = \frac{1}{n-1} Tr_A(\rho_A^n) \ .$$

(The von-Neumann entropy is recovered from the $n\to 1$ limit, if the limit exists.) The S_n can be calculated from the partition function of the theory on an n-sheeted covering of $\mathbb{R}^2$:

$$C_n := \cup_{i=1}^n \mathbb{R}^2_{(i)}$$

such that for any field ϕ, it takes the same values for all $\mathbb{R}^2_{(i)}$ away from A and on A we glue ϕ_i and ϕ_{i+1} in a cyclic fashion. This construction can be regarded technically as some twist field correlation function in a symmetric orbifold theory. Less abstractly, near the points a,b we simply have a conical singularity with the total angle in the range $\theta\in[0,2\pi n]$. The space C_n constructed above can actually be mapped conformally to the one with $A=[0,\infty]$.[16] Then the C_n space associated to it is manifestly equivalent to the ordinary complex plane after a transformation

$$z = w^{1/n} \ , \qquad w\in C_n \ , z\in\mathbb{R}^2 \ .$$

[16] **Exercise 5:** Show that in two dimensions any interval can be conformally mapped to the half line.

Now, using the fact that the holomorphic energy-momentum tensor does not transform covariantly, using the explicit expressions (19),(20) we find that the conical defects at a,b behave as primary operators of dimensions

$$\Delta_n = \frac{c}{24}(1-1/n^2)\ .$$

The Renyi entropy, being just the partition function on C_n, is therefore given by

$$S_n \sim \frac{1}{n-1}|a-b|^{-c/6(n-1/n)}\ . \tag{134}$$

In particular, the $n \to 1$ limit exists and gives

$$S_{von-Neuman} \sim c\log(|a-b|)\ . \tag{135}$$

The analysis of the case that A consists of a disjoint union of intervals is far more complicated and we will not discuss it here.

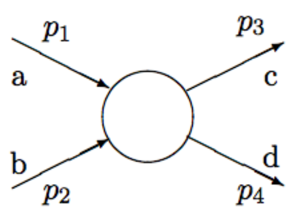
There is also a very interesting reinterpretation using the entanglement-entropy of the interval. when the interval $[a,b]$ is very short compared to the mass scale, then we recover (135) with $c \to c_{uv}$. When the interval becomes very long compared to the mass scale, then we have (135) with $c \to c_{ir}$. In between there is some interpolating functions. Using information-theoretic properties of entropy, it is possible to show directly [30] from this point of view that $c_{uv} > c_{ir}$. So it would seem that the monotonicity of the renormalization group flow in two dimensions is essentially equivalent to some properties of the von-Neumann entropy.

Chapter 9
S-Matrix Theory

Let A be the scattering amplitude, which is assumed to be analytic other than singularities corresponding to physical processes. Unitarity has a simple physical meaning: the sum of probabilities of all processes which are possible at a given energy is equal to unity, $SS^\dagger = 1$. If $S = 1 + iA$, then Representing the amplitude A as the sum of its real and imaginary parts, $A = \Re A + i\Im A$, the unitarity condition takes the form

$$2\Im A = AA^\dagger \ .$$

For 2-2 scattering we define the standard Mandelstam variables



$$s = (p_1 + p_2)^2 \ , \qquad t = (p_1 - p_3)^2 \ , \qquad u = (p_1 - p_3)^2 \ .$$

They are actually not independent

$$s + t + u = \sum_{i=1}^{4} m_i^2 \ .$$

At the center of mass frame, for identical particles of mass m, the scattering angle is $cos\theta = 1 + \frac{2t}{s-4m^2}$.

Here we will make some comments about an S-matrix that is saturated by stable resonances, namely, that only tree-level diagrams are included. Such an S-matrix cannot be by itself unitary, but it becomes exact at large N in theories like QCD. The consequences of unitarity are much easier to analyze in this case. On the other hand, due to the intrinsic interest in large N QCD and related theories such as String Theory, this still allows to study interesting questions.

If we imagine now scattering identical scalar particles then we have a scattering amplitude $A(s,t)$ which satisfies a couple of requirements:

- $A(s,t) = A(t,s)$. This is just permutation symmetry between the particles.
- For a fixed t we have a meromorphic function in s with simple poles at $\{p_n\}$ with real p_n. If there are no tachyons then all the p_n are non-negative.
- Unitarity: $Res_{s=p_n}A(s,t) = \sum_k f_k P_k(cos\theta)$ with $f_k \geq 0$ and at the residue we can use $cos\theta = 1 + \frac{2t}{p_n - 4m^2}$. The P_k are Legendre polynomials in four dimensions, and in general dimension we should use ${}_2F_1\left(-j, j+D-3, \frac{D-2}{2}, \frac{1-\cos(\theta)}{2}\right)$. Therefore, the residues are some polynomials in t with positive coefficients in the appropriate basis of spherical harmonics.

The physical interpretation of the residue is that it tells us which intermediate particles of mass squared p_n appear. If the Legendre polynomial P_l is present in the expansion of the residue at $s = p_n$ then a particle of spin l and mass $\sqrt{p_n}$ is exchanged.

It is actually very easy to find amplitudes that obey all of these axioms. A simple example is

$$A(s,t) = \frac{1}{s-m^2} + \frac{1}{t-m^2}$$

which describes a single massive scalar resonance.

The theory of scattering amplitudes becomes much more interesting and much more constrained if we demand that the amplitude decays at large s, which means that the UV is very soft. This is what happens in String Theory and perhaps in many other examples (why?!)

So let us throw another requirement into the pot

$$\exists t_0 \;\text{ such that }\; \lim_{s\to\infty} A(s,t_0) \to 0\ .$$

Now it appears extremely difficult to solve all the constraints. (this requirement is like saying that the Regge function crosses zero at some point – why does it have to? Maybe instead of this requirement we should demand Regge, which sounds a little more physical.)

One can immediately prove that infinitely many resonances are necessary in order to satisfy this constraints of decaying amplitude.

Proof: $\lim_{s\to\infty} A(s,t) \to 0$ implies that we can deform a small contour γ around s'

$$A(s',t) = \frac{1}{2\pi i}\int_\gamma A(s,t)\frac{ds}{s-s'}\ .$$

to infinity, while avoiding the poles

$$A(s',t) = \frac{1}{2\pi i}\int_{\mathbb{R}+i\varepsilon} A(s,t_0)\frac{ds}{s-s'} - \frac{1}{2\pi i}\int_{\mathbb{R}-i\varepsilon} A(s,t_0)\frac{ds}{s-s'}\ .$$

We therefore only need to know $\Im A$ on the real axis, which is given by

$$\Im A(s,t) = \sum_n H_n(t)\delta(s-p_n)$$

with $H_n(t)$ a non-negative finite sum of Legendre polynomials of $\cos(\theta) = 1 + \frac{2t}{p_n - 4m^2}$.

Hence, we find that

$$A(s,t) = \sum_n \frac{H_n(t)}{s-p_n} \ . \tag{136}$$

This decomposition converges for those t that satisfy $\lim_{s\to\infty} A(s,t) \to 0$. Typically these are negative t's. Indeed, for negative t the Legendre polynomials are oscillating and the sum is better behaved.

From the representation (136) we see that the amplitude can satisfy

$$A(s,t) = A(t,s)$$

only if there are infinitely many terms in the expansion. Otherwise, the function is only a polynomial in t without singularities.

In fact, one can prove that for every k that exists at least once, infinitely many $H_n(t)$ must have a nonzero component of the kth Legendre polynomial. Physically, this means that we have infinitely many particles of every spin that exists at least once. This already starts smelling like Regge trajectories, Hagedron densities and many other nice things we expect from Large N gauge theories and String Theories.

Proof: We begin from the standard decomposition explained above (now performed in the t-channel):

$$\sum_i \sum_{m_i} \frac{C(m_i,i)}{t-m_i^2} P_i\left(1 + \frac{2s}{m_i^2 - 4m^2}\right) \ .$$

Unitarity is the statement that $C(m_i,i) \geq 0$. We assume that for $s<0$ this expansion converges. In particular we will be interested in the kinematical regime

$$s < 0 \qquad \text{and} \qquad t \in (0, 4m^2 - s)$$

In this case, the scattering angles θ is real (let's discard the states with mass$< 2m$ for the simplicity of the argument).

We can now project $A(s,t)$ to spin j in the s-channel. From the s-channel decomposition of $A(s,t)$ we get

$$\int_0^{4m^2-s} \frac{dt}{4m^2-s} A(s,t) P_j\left(1 - \frac{2t}{4m^2-s}\right) = \frac{a_j(s)}{2j+1}$$

If we only have finite many masses at spin j then $a(s,j)$ is just a rational function with poles at the position of these masses and positive residues. On the other hand, if we plug into the same integral, the t-channel decomposition we get

$$\frac{2}{4m^2-s}\sum_i\sum_{m_i}C(m_i,i)P_i\left(1+\frac{2s}{m_i^2-4m^2}\right)Q_j\left(1+\frac{2m_i^2}{s-4m^2}\right)$$

where we integrate slightly above the real axis and

$$Q_j(x)=q_j(x)+\frac{1}{2}P_j(x)\log\frac{x+1}{x-1}$$

Here, $q_j(x)$ is a known polynomials of degree $j-1$ (it can be read from P_j by demanding that at large z, Q decays as $1/z^{j+1}$). The function $Q_j(x)$ is the other solution to the partial wave differential equation that defines the Legendre polynomials.

Let m_* to be the first mass that contains a particle of spin j for which $4m^2-m_*^2<0$. For $4m^2-s<m_*^2$, we do not reach the logarithmic branch point of $Q_j(1+\frac{2m_*^2}{s-4m^2})$ yet. If we now take the limit

$$s\to s_*\equiv 4m^2-m_*^2$$

then the terms in the t-channel decomposition with mass m_* contribute

$$(-1)^j\log(s-s_*)\frac{1}{m_*^2}\sum_{m_i=m_*}C(m_i,i)(-1)^i+[\text{regular}] \tag{137}$$

while the other contributions are regular at this point. The coefficient of the log in (137) is positive by unitarity. Therefore, we conclude that for any spin j carried by a particle there are infinite many particles of that spin.

There are not many explicit examples of amplitudes that satisfy all of these constraints. One well known example is the Veneziano amplitude, where our external particles are tachyons, with mass squared -1 in some units.

Consider

$$A(s,t)=-\frac{\Gamma(-s-1)\Gamma(-t-1)}{\Gamma(-s-t-2)}$$

It has simple poles at $s=-1,0,1,2,...$ and $t=-1,0,1,2,....$ Let us consider the residue at $s=n$ where $n\in\{-1,0,1,2...\}$.

$$A(s\to n,t)=\frac{Pol_{n+1}(t)}{s-n}\ ,$$

where up to an overall coefficient $Pol_{n+1}(t)=(-1)^{n+1}\frac{\Gamma(-t-1)}{\Gamma(-n-t-2)}$ is a polynomial of degree $n+1$.

We can always decompose

$$Pol_{n+1}(t)=\sum_{l=0}^{l=n+1}a_{n+1}^lP_l\left(1+\frac{2t}{n+4}\right)$$

Claim: $a_l^n\geq 0$ for all l,n.

Instead of using the Legendre polynomials, one can also try their natural generalization to $SO(D-1)$ polynomials. (Legendre polynomials appear for $D=4$.) These are defined by

$$P_j(z) = {}_2F_1\left(-j, j+D-3, \frac{D-2}{2}, \frac{1-z}{2}\right) .$$

(They can be also defined as the solution to some differential equation.) The claim $a_l^n \geq 0$ remains true for all $D \leq 26$. This is why 26 is the critical dimension of string theory.

If there is a direct proof of this result, it would be invaluable. The only general proof that exists (as far as I know) relies on physical arguments having to do with the string worldsheet theory.

Explicit Check of $n=-1,0,1$

A useful representation of the Legendre polynomials that are relevant for scattering in D space-time dimensions is

$$\left(\frac{1}{1-2\eta x+\eta^2}\right)^{\frac{D-3}{2}} = \sum_k \eta^k P_k^D(x)$$

From this we learn that

$$P_0^D(x) = 1 , \qquad P_1^D(x) = 2\alpha x , \qquad P_2^D(x) = 2x^2(\alpha^2+\alpha) - \alpha ,$$

with $\alpha = \frac{D-3}{2}$.

On the other hand, $Pol_{n+1}(t) = (t+2)(t+3)\cdots(t+n+2)$. In order to express it as a function of $\cos\theta$, we use the change of variables $t = \frac{n+4}{2}(\cos\theta - 1)$. The case of $n=-1$ is just $Pol_0(t) = 1$ so it is obviously unitary. The case of $n=0$ we have $Pol_1(\cos\theta) = 2\cos\theta$ which is proportional to $P_1^D(x)$ with a positive coefficient. Finally, the case of $n=1$ is

$$Pol_2(\cos\theta) = \frac{25}{4}\cos^2\theta - \frac{1}{4} .$$

In order to decompose that in terms of P_2^D and P_0^D with only positive coefficients, it has to be true that the ratio of the x^2 and constant term in P_2^D is smaller than 25, hence, $\alpha+1 \leq 25/2$ which means $D \leq 26$, the critical dimension of string theory. For $D > 26$, the spin 0 mode in the $n=1$ pole does not have a positive coefficient so it is a ghost.

The Veneziano amplitude appears to be really special. Are there any interesting deformations of it? Do we always get linear Regge trajectories? Hagedorn densities?

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